Dynamic Bayesian Persuasion with a Privately Informed Receiver*

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Abstract

We study a *dynamic Bayesian persuasion* framework in a finite horizon setting consisting of a *seller* and a *buyer*. The seller wishes to persuade the buyer to buy a durable good at a given price by providing information about its relevance (match quality). The buyer has *private information* about his valuation for a good match and we study *optimal dynamic information policies* employed by the seller in equilibrium. For a *fixed horizon*, we show that the seller always provides signals which truthfully convey a good match but may garble a bad one. Moreover, if the good is not bought in the first stage, the seller provides information which improves over time. The agents always interact for a fixed amount of time within which a purchase decision is made. The length of this interaction remains fixed even for long horizons and depends only on the prior on the buyer's valuation. As the *horizon goes to infinity*, the bulk of the information about match quality is provided in the first period. This allows the buyer to extract a large amount of information from the seller at the beginning of their interaction. Even a slight probability of the buyer being difficult to convince facilitates close to full disclosure immediately.

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1 Introduction

In many settings of economic interest, the role of information is crucial in influencing actions undertaken by agents. In addition to incentives that emerge from factors that directly influence one's own payoffs, the quality and nature of information available to agents about payoff-relevant state variables plays an equally important role in influencing their decisions. Typically, incentives of this nature arise in the following manner : optimal actions of an agent depend on his belief or opinion about elements of uncertainty within the decision making framework; the beliefs themselves are guided by the precise form of information available. One may view the class of all such situations as being divided into two broad categories : *exogeneous* and *endogeneous* presence of information. In the former category, nature controls all information available within the environment whereas in the latter, the extent to which information is provided lies in the hands of an active strategic agent with non-trivial objectives of his own. It is the latter class of situations to which this paper pertains.

Consider the following scenario consisting of two parties : a *seller* and a *buyer*. The seller wishes to sell a durable good (for eg. a car, house or electronic device) to the buyer at a given price. The good can either be a *good* or a *bad* match for the buyer. If the match is indeed good, then the buyer strictly prefers to buy and wishes to refrain from buying if the match is bad. On the other hand, the seller only wants for the good to be bought and is apathetic to the buyer being ill-matched. Ex-ante, both agents are uninformed of the true match quality i.e they are both *symmetrically uninformed*. Moreover, the buyer is *privately informed* about his valuation for a good match and has either a *high* or a *low* valuation. The seller possesses the ability to provide verifiable information about the durable good over time. This in turn reveals information about the match quality and affects the chances of the buyer purchasing. The seller's objective is then to optimally choose the disclosure of such information over time in order to maximise the chances of a purchase.

Let us discuss the above scenario in the context of an example. Consider a situation concerning the purchase of a car at a car dealership. The car salesman (seller) wishes to persuade a potential customer (buyer) to buy a given car at a stated price. The car in question may either match (good) or not match (bad) the customer's preferences. A good match here could be a satisfaction of a set of characteristics that the customer deems valuable. This could be based on a set of factors such as safety, reliability and performance. For example, the customer may want to know the effectiveness of the warning system for

lane departure for safety concerns. The customer privately knows how much he values a car (buyer valuation) that is a good match for him. This is determined by the nature of personal use of the car by the customer. Certain types of customers may wish to use public transportation on certain days and may drive less often but would nonetheless wish to buy a car that matches their preferences. The salesman wishes to persuade the customer to buy by providing verifiable information (for eg. by allowing the customer to test drive the car) over time with the objective to maximise the chances of the customer buying.

We analyse the above scenario by studying a *dynamic game in finite horizon* whose chief elements comprise the features highlighted above. Every period, the seller chooses a *signal* or *Blackwell experiment* that is informative about the match quality. The choice of experiment and its outcome is publicly observed upon which the buyer chooses to either buy or not buy the durable good. A decision to buy is irreversible and ends the game. If the good is not bought, play proceeds to the next stage wherein the seller chooses another experiment followed by the buyer's purchasing decision. We assume that in choosing experiments, the seller lacks long term commitment power and faces no costs in providing information. We are interested in studying Perfect Bayesian Equilibria of the resulting game by restricting attention to Markovian Strategies. Hence, actions of agents at any stage depend solely on the belief about the match quality, the belief about the buyer's valuation (type) and number of periods left in the game.

Central to our analysis is the study of incentives of the two agents. Since the buyer wishes to buy only under a good match, the seller has an incentive to present evidence in a way that deems the good match more likely. On the other hand, since the buyer is a rational Bayesian, he understands that any experiment that generates an outcome highly favouring the good match can also generate an outcome favouring the bad match. This imposes a constraint on the extent to which the seller can maneuver beliefs about the match quality and consequently chances of inducing a purchase. Given these factors, what is then the best way for the seller to provide information? Consider now the incentives of the buyer. At any stage, he faces a trade-off between two options : i) buying the durable good at the current belief about match quality ii) not buying in the current period and accumulating further information about match quality. Hence, he would be inclined to buy only if the expected benefit from the purchase is atleast as much as the value of information he would obtain by deferring his decision to the future.

Now, such incentives would of course be different for the high and low valuation buyer.

The low valuation buyer would always be less likely to buy which allows the seller to learn about the buyer's valuation from his actions. In particular, if the seller observes that the good has not been bought, he would be more likely to believe that it was the low valuation buyer who chose to not buy. Hence, information is communicated two ways : i) The seller provides information about match quality ii) The buyer provides information about valuation via purchasing decisions. How much information is exchanged in equilibrium and how long does communication take place? Is information conveyed gradually over time or in bulk? To what extent do the answers to questions posed so far depend on the length of the horizon? The following section highlights the main results.

Main Results : When the horizon is *fixed*, the optimal dynamic information policy involves the use of two-message experiments (meaning experiments use a good or a bad signal) which only garble the bad match and truthfully convey the good match. If the good is not bought in the first period, then future experiments provided by the seller become *increasingly informative* in the Blackwell sense. The number of *rounds of communication* is completely determined by the prior on the buyer's valuation and is independent of horizon length. If this prior is above a threshold, then conditional on only good signals being realised, the valuation of the buyer is eventually learnt with probability one.

As the horizon *goes to infinity*, the above features remain preserved and additionally two important new features arise. Firstly, the *bulk* of the information about the payoff-relevant state is provided in the first period. Secondly, a *reputation effect* emerges: even a slight probability of being extremely hard to convince (the buyer has a valuation very close to the fixed price) facilitates close to full disclosure immediately.

Related Literature : This paper pertains to the recent and growing literature on *Bayesian Persuasion* and *Information Design*. The literature studies models of strategic transmission of information between two parties namely a *Sender* and a *Receiver*. The objective of the Sender is typically to induce desired actions by optimally disclosing information. In our context, the seller is the counterpart of the Sender and the buyer the counterpart of the Receiver. The desired action is a decision to buy and the seller can persuade the buyer to undertake it only by structuring and supplying information. Kamenica-Gentzkow (2011[17]) Rayo-Segal (2010[24]) and Kolotilin et. al (2015 [21]) consider static environments concerning persuasion. These papers consider environments with richer state spaces and preferences compared to the present work. Kamenica-Gentzkow (2011[17]) consider a setting where the Receiver's payoffs are known and provide important method-

ological approaches to study the problem. Rayo-Segal (2010[24]) and Kolotilin et. al (2015 [21]) study environments where the Receiver has private information about tastes. The former study *public persuasion* mechanisms where the Sender, being uncertain of the Receiver's type, chooses optimal information to disclose. The latter compare such mechanisms to *private persuasion* mechanisms where the Sender first asks the Receiver to report his type depending on which information is provided. In the framework considered in this paper, the dynamic game involves the seller engaging in public persuasion every period. Further, he lacks long term commitment power and can make use of the information he obtains from the buyer about his valuation in subsequent rounds (this will not arise in a static framework).

In dynamic settings concerning Bayesian persuasion and information design, Au (2015)[1] considers a model in which the Receiver is privately informed about his payoffs. In contrast to the analysis considered here, Au (2015) considers an infinite horizon model with discounting and a continuum of types for the Receiver. The approach adopted by the paper involves interpreting the problem as a bargaining problem with one-sided offers and one-sided incomplete information. Using this formulation, the paper derives a full disclosure result as consequence of coasian dynamics along with other qualitative features of equilibrium. We adopt a similar approach in the present framework and are able to derive a closed form characterisation of equilibrium in Markovian strategies. This allow us to perform comparative statics and makes all equilibrium characteristics transparent. Ely (2015 ([10])), Che-Horner(2015 [9]), Renault-Solan-Vielle (2014 [25]) study and characterise optimal dynamic information policies when the Receiver's payoffs are known. These papers focus on environments where payoff relevant states of nature may change over time and the Sender has long term commitment power. Ely (2015) and Renault-Solan-Vielle (2014) provide methods to approach a broad class of persuasion problems with a myopic agent (Receiver). Ely (2015) further studies the case of a long run agent and also dynamic information design for multiple agents. Che-Horner (2015) study information design in a context of social learning where an internet platform designs the optimal recommendation system for consumer experimentation with an experience good. Horner-Skrzypacz(2016) and Orlov-Skrzypacz-Zryumov (2016) consider dynamic bayesian persuasion problems where the Sender lacks long term commitment power. The former work considers a setting where an agent (Sender) wishes to convince a firm (Receiver) to hire him at a wage by undergoing tests about his competence. The latter consider a setting involving a firm (Sender) and a regulator (Receiver). The firm produces a drug which can be recalled from the market by the regulator based on information ob-

tained from an exogenously given source of news and additional tests conducted by the firm. The central question addressed is how the firm may supply information to persuade the regulator to not choose the recall option. Ely-Frankel-Kamenica(2015)[11] study an environment with a fixed state where preferences over non-instrumental information are exogeneously given and a principal (Sender) chooses an optimal information policy (belief martingale). The crucial point of distinction between the papers discussed above is the fact the buyer in the current work is privately informed about his valuation. This feature yields very strong implications for behaviour of the two parties and differs drastically from behaviour when the valuation is known. Morever, as we shall see, this difference persists even when uncertainty about the buyer's valuation is ex-ante small. Lastly, papers due to Bergemann-Morris(2016)[3], Taneva (2016)[23] and Mathevet-Perego-Taneva (2016)[19] study the issue of information design in games i.e the interaction between a Sender and multiple Receivers.

The reputation effect studied in this work relates it to dynamic models of reputation studied by Kreps-Wilson (1982)[15], Milgrom-Roberts (1982)[20], Kreps-Milgrom-Roberts-Wilson (1982)[16], Fudenberg-Levine (1989)[12]. These papers show that a slight probability of appearing tough can facilitate extracting a payoff much higher than one would have otherwise got. In the present work, the low valuation buyer is the counterpart of the commitment type in these models. Toughness here translates to the notion of being "hard to convince" in the sense that low valuation buyers need more information in order to purchase. Note however, that all players in the present framework are long run rational players. This is a point of distinction with these papers which include a sequence of short run players and commitment types.

Lastly, certain qualitative features of our analysis bear resemblance with features observed in models of bargaining with incomplete information as in Fudenberg-Levine-Tirole (1985)[13], Chatterjee-Samuelson(1987)[7] and Chatterjee-Samuelson(1988)[8]. Fudenberg -Levine-Tirole (1985) consider a model of bargaining with one-sided incomplete information where an uninformed seller repeatedly makes price offers to a buyer who is privately informed about his valuation. Chatterjee-Samuelson(1987)[7] and Chatterjee-Samuelson(1988)[8] study environments with two-sided incomplete information with alternating offers. It is possible to draw a correspondence between these models and the framework considered here. The information provided by the seller is publicly observable and hence both the buyer and the seller are symmetrically informed about the match quality. The buyer is additionally privately informed of his valuation. This can be viewed

as a one-sided incomplete information environment where the uninformed party (seller) chooses to offer experiments informative about the relevance of a durable good in order to persuade an informed party (buyer) to make a purchase. The private information (valuation) is critical in determining gains from trade and certain types of buyers may wish to pool with other types to obtain better offers. In the present context, the high valuation buyer has an incentive to not buy, signalling that his valuation is low to obtain better information.

Outline of Paper : The paper is organized as follows. Section 2 introduces the framework describing central elements of the environment considered and provides the anaylsis of the one-shot game. Section 3 provides a treatment of the two-period game. Section 4 analyses the T-period game where all the main results lie.

2 Framework

We consider a dynamic framework where time is discrete and the horizon is finite i.e $t \in \{1, 2, ..T\}$ where $T \in \mathbb{N}$ denotes the horizon length. The environment consists of two agents : A *buyer* and a *seller*[1]. The seller wishes to sell a durable product at a given price[2] $p > 0$ to the buyer whose decision to buy depends on whether or not the good is relevant for purchase. The product can either be a *good* or a *bad* match for the buyer. The *match quality* is denoted by $\omega \in \Omega := \{G, B\}$ where $\omega = G$ denotes a good match and $\omega = B$ denotes a bad match[3]. The match quality is unknown to both the buyer and the seller but they assign prior probability μ_0 on $\omega = G$. Additionally, the buyer has private information about his *valuation* $v \in \mathcal{V} := \{v_1, v_2\}$ for a good match. We assume that $v_1 > v_2 > p$ i.e the buyer of *type* v_1 places a higher value for a good match and both buyer types benefit from purchase if the match quality is indeed good. Finally, the seller assigns prior probability π_0 on $v = v_1$ and we assume ω and v are drawn independently.

At any point in time, the buyer either takes an irreversible decision to *buy* or chooses to *not buy* and potentially defer his purchasing decision. Hence, he chooses an action

[1]While we introduce the framework in a setting involving the interaction between a buyer and seller, all results generalise to a broader class of dynamic Bayesian persuasion problems. We provide an explanation in the discussion section

[2]Here we do not model the mechanism through which the price is determined and assume that agents take the price as given.

[3]We shall occasionally refer to elements of Ω as *states of nature* since they represent payoff relevant elements of uncertainty. The values G and B will be referred to as the *good* and *bad* state respectively.

$a \in A := \{b, nb\}$ where $a = b$ refers to a buying decision and $a = nb$ refers to a decision to not buy. At any stage, if t periods are left, the buyer's decisions yield the following payoffs

	G	B
b	$(v-p)t$	$(-p)t$
nb	0	0

Hence, if a buyer (v) makes a decision to buy and the match quality is good, then he obtains a lifetime utility of $(v-p)t > 0$. However, if the match quality is bad, then his lifetime payoff equals $(-p)t < 0$. A decision to not buy yields a payoff equal to 0 for the current period. The seller gets a payoff equal to $pt > 0$ if the buyer chooses to buy irrespective of the true match quality.

The payoffs described above can be interpreted in two ways. Under the first interpretation, we view the lifetime utilities of both agents as a total payoff accrued from a constant stream of utilities for the rest of time. For example, for the buyer, purchasing the product of good match quality yields a constant stream of utilities equal to v (likewise a constant stream of zeroes if the match is of bad quality). In addition, the buyer pays the constant fixed price $p > 0$ to the seller over time which gets deducted from the utility from holding the durable good. Under the second interpretation, the multiplicative term t can be viewed to reflect a time preference for obtaining a unit of utility. For both agents, the value of obtaining a certain outcome that yields utility x when t periods are left equals xt[4]. The two interpretations are closely related and the reader is encouraged to view the present analysis under either of them.

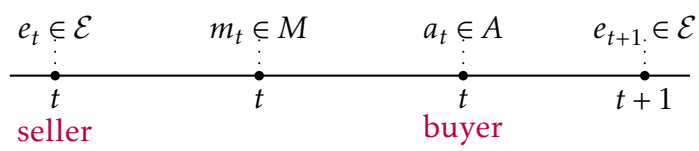
Finally, we impose the assumption that $\mu_0 < \frac{p}{v_1}$. This means that in the absence of any further information about the match quality both buyer types would choose to never buy. In order to see this, observe that if at any stage the buyer holds a belief μ about the match being good and no further information is given, then a buyer of type v_i would choose to buy at that stage if and only if $\mu \geq \frac{p}{v_i}$. Since by definition $v_1 > v_2$, this inequality fails to hold under the given assumption for both buyer types. For notational convenience, in what follows, we shall define these *thresholds* to be $\mu_i^* := \frac{p}{v_i}$. Further, we define the following quantity :

$$\sigma := 1 - \frac{\mu_1^*}{\mu_2^*} \tag{1}$$

[4]See for example Fishburn-Rubinstein (1982)

We shall call σ the *threshold gap* since it measures how far apart the thresholds for the buyers are. Note that if the two thresholds μ_1^* and μ_2^* are close to each other, then σ is close to zero. If they are very far apart, then the threshold gap σ is close to one. These objects will play a very important role in our analysis and studying the behaviour of agents.

Description of the game : The dynamic game between the seller and the buyer is as follows. Every period (if the good has not already been bought), both agents play a two-stage extensive form game : 1) The seller first chooses what information to provide to the buyer i.e the seller chooses a Blackwell experiment or signal $e \in \mathcal{E} := \{e' | e' : \Omega \to \Delta(M)\}$[5]. 2) The buyer observes the experiment chosen and its outcome and chooses an action $a \in A = \{b, nb\}$. A decision to buy ends the game and a decision to not buy allows the game to proceed to the next period wherein the same two-stage extensive form is again played. Play continues either till the point when the buyer decides to buy or when a decision to buy is never made and the end of the horizon is reached. The illustration of the two-stage extensive form at a given time period is provided below :



A game with horizon length T and initial prior beliefs (μ_0, π_0) will be denoted as $\hat{G}^T(\mu_0, \pi_0)$. We now define *strategies* for each agent in the described game.

Definition 1. *In the T-period game, a strategy for the seller is defined as a function* $\hat{\sigma}_S : \bigcup_{t=0}^{T-1} (\mathcal{E} \times M)^t \to \Delta(\mathcal{E})$ *and a strategy for the buyer (v) is defined as a function* $\hat{\sigma}_R^v : \bigcup_{t=0}^{T-1} (\mathcal{E} \times M)^t \times \mathcal{E} \times M \to \Delta(A)$.

In order to define equilibrium, we must define *beliefs* for each agent. Since actions of both agents are observable, only beliefs about the match quality and buyer type need to be specified. A strategy profile when paired with a system of beliefs for agents is said to be an *assessment*. We define these formally below :

Definition 2. *In the T-period game* $\hat{G}^T(\mu_0, \pi_0)$*, a system of beliefs for the seller is represented by* $\hat{\pi}_S : \bigcup_{t=0}^{T-1} (\mathcal{E} \times M)^t \to \Delta(\Omega \times \mathcal{V})$ *and a system of beliefs for buyer v is represented by* $\hat{\pi}_B^v$:

[5]Here M is a given countable message space for signal realisations.

$\bigcup_{t=0}^{T-1}(\mathcal{E}\times M)^t\times\mathcal{E}\times M\to\Delta(\Omega)$. *Any tuple* $<(\hat{\sigma}_S,\hat{\sigma}_B^{v_1},\hat{\sigma}_B^{v_2});(\hat{\pi}_S,\hat{\pi}_B^{v_1},\hat{\pi}_B^{v_2})>$ *consisting of a strategy profile and a system of beliefs for agents is called an assessment.*

The notion of equilibrium we shall use here will be Perfect Bayesian Equilibrium. The equilibrium concept requires that agents choose optimal actions given the system of beliefs with the beliefs themselves being derived from play induced by strategies by using Bayes rule wherever possible. In what follows, we shall impose the following requirement over the system of beliefs : If a seller deviates from his strategy and chooses a different experiment, then both agents update beliefs about the match quality (state) according to the experiment chosen and its outcome. This requirement is natural since the choice of experiment is not in itself informative about ω and this choice along with the experiment's outcome is publicly observable. Hence, if an experiment distinct from what the seller's strategy prescribes is indeed chosen, both agents would take into account probabilities of outcomes (messages) corresponding to the chosen experiment. We shall also restrict attention to *Markovian strategies* which depend solely on the beliefs of the agents $(\mu,\pi)\in\Delta(\Omega\times\mathcal{V})$ and number of periods t left in the game.

2.1 One-shot game

In this section, we analyse the one-shot game between the buyer and the seller i.e when $T=1$. The seller provides an experiment which generates a posterior belief on Ω depending on the outcome of the experiment. Depending on the posterior belief, the buyer chooses whether to buy or not after which the game ends. Note that any choice of experiment leads to a distribution of posteriors $\tau\in\Delta(\Delta(\Omega))$ with mean equal to the prior μ_0 i.e $\mathbb{E}_\tau(\mu')=\mu_0$[6] (See for example Kamenica-Gentzkow (2011[17])).

If the seller provides a posterior $\mu<\mu_1^*$, then both buyer types would choose to not buy resulting in an expected payoff of 0 to the seller. Similarly, if the posterior provided is $\mu\geq\mu_2^*$, both buyer types would choose to buy giving the seller an expected payoff of q. Now if the posterior is $\mu\in[\mu_1^*,\mu_2^*)$, then high valuation buyer v_1 would choose b and the v_2 buyer type would choose to play nb resulting in an expected payoff of $\pi_0 p$ for the seller. Hence, the seller's value function (expected payoff as a function of posteriors provided)

[6]Note since Ω consist of only two elements we represent elements in $\Delta(\Omega)$ by the probability they assign on $\omega=G$.

is the following :

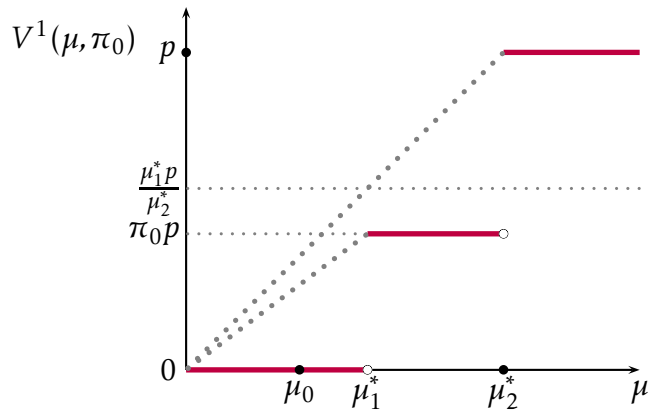
$$V^1(\mu) = \begin{cases} p & \mu \geq \mu_2^* \\ \pi_0 p & \mu \in [\mu_1^*, \mu_2^*) \\ 0 & \mu < \mu_1^* \end{cases}$$

Hence, in this case, the optimal disclosure policy of the seller would be dependent on the value of π_0 and can be obtained using the concavification of the function V^1 described above (see for example Kamenica-Gentzkow (2011), Aumann-Maschler(1995)). One can now show that if $\pi_0 < \frac{\mu_1^*}{\mu_2^*}$, then the optimal signal would provide the posterior μ_2^* with probability $\frac{\mu_0}{\mu_2^*}$ and a posterior of 0 with probability $1 - \frac{\mu_0}{\mu_2^*}$. We denote the experiment as $(0, \mu_2^*)$. seller gets an expected payoff of $\frac{\mu_0}{\mu_2^*}p$. If $\pi_0 > \frac{\mu_1^*}{\mu_2^*}$, then the optimal signal would provide the posterior μ_1^* with probability $\frac{\mu_0}{\mu_1^*}$ and a posterior of 0 with probability $1 - \frac{\mu_0}{\mu_1^*}$. The experiment is denoted as $(0, \mu_1^*)$ and results in the seller getting an expected payoff of $\pi_0 \frac{\mu_0}{\mu_1^*} p$. If $\pi_0 = \frac{\mu_1^*}{\mu_2^*}$, then the seller would be indifferent between $(0, \mu_1^*)$ and $(0, \mu_2^*)$. Hence, the set of optimal experiments would be the set of all convex combinations of $(0, \mu_2^*)$ and $(0, \mu_1^*)$ [7]. Note we can write $\frac{\mu_1^*}{\mu_2^*}$ in terms of the threshold gap equals $1 - \sigma$. An optimal strategy for the seller is :

$$\sigma_S^1(\mu_0, \pi_0) = \begin{cases} (0, \mu_2^*) & \text{if } \pi_0 < 1 - \sigma \\ \lambda(0, \mu_1^*) + (1 - \lambda)(0, \mu_2^*) & \text{if } \pi_0 = 1 - \sigma \\ (0, \mu_1^*) & \text{if } \pi_0 > 1 - \sigma \end{cases}$$

The following diagram illustrates the value function for the case $\pi_0 < 1 - \sigma$. Note that the value of μ_0 on the line joining $(0,0)$ and $(\mu_1^*, \pi_0 p)$ is strictly lower than the corresponding value of the line joining $(0,0)$ and (μ_2^*, π). This shows that the seller strictly prefers experiment $(0, \mu_2^*)$ over $(0, \mu_1^*)$. Morever, the value $(0, \mu_2^*)$ is greater than from any other distribution over posteriors. The corresponding dotted line from 0 to μ_2^* is precisely the value of concavification of the function v on the region $[0, \mu_2^*]$.

[7] Formally, for any two distributions over posteriors $\tau_1, \tau_2 \in \Delta(\Delta(\Omega))$ with mean μ_0 and $\lambda \in [0,1]$, the distribution induced by $\tau^\lambda := \lambda\tau_1 + (1-\lambda)\tau_2 \in \Delta(\Delta(\Omega))$ has mean equal to μ_0. Hence, τ^λ is induced by some experiment $e \in \mathcal{E}$. Morever, for any bounded measurable function $f : \Delta(\Omega) \to \mathbb{R}$, the expectation $\mathbb{E}_{\tau^\lambda}[f]$ is linear in λ.



3 Two Period Interaction

Consider now the same setting with two stages of interaction i.e. when $T = 2$. The match quality ω and the private type v of the buyer are drawn independently at the beginning of the game followed by a two period game. In the first period, the seller chooses an experiment and the buyer chooses whether to buy or not buy. If a buying decision is made, the game ends. If the buyer chooses not to buy, the game proceeds to the second period wherein the seller chooses another experiment and the buyer makes a purchasing decision. In this situation, once a seller chooses an experiment in the first period and a posterior is generated, the buyer faces a trade-off. He can either choose to buy and obtain an expected payoff from possessing the good at the given posterior or he could choose to not buy and collect further information in order to make a more informed purchase decision in the second period. Since holding the good is valuable under a good match, waiting for better information is inherently costly making the trade-off non-trivial.

Since we are interested in PBE in Markovian strategies, actions and continuation values depend solely on the triple $(\mu, \pi, t) \in [0,1]^2 \times \{1,2\}$. In order to ascertain the choice of experiment made by the seller in the first period, it suffices to obtain equilibrium continuation values $V^2(\mu, \pi)$. Essentially, similar to the value function considered in the one-shot case, $V^2(\mu, \pi)$ corresponds to the value the seller would obtain upon providing a posterior μ in the first period. The concavification of $V^2(., \pi)$ is then used to derive the optimal experiment chosen in the first period. Before proceeding, it shall be helpful to introduce some notation. At a given belief μ about match quality, the tuple (μ', μ'') where $\mu' \leq \mu \leq \mu''$ would represent the information structure which provides posterior μ' with probability q and posterior μ'' with probability $(1-q)$. Notice that q is uniquely determined by the martingale condition $q\mu' + (1-q)\mu'' = \mu$ if $\mu' < \mu''$ and the case $\mu' = \mu''$ represents no information. Hence, the notation (μ', μ'') conveys the underlying information structure

without any loss of generality. We are now ready to analyse the game.

We first argue that for $\mu < \mu_2^*$, in the subsequent game, the buyer of type v_2 would choose to not buy (with strict preference) in any equilibrium. Suppose the seller observes a decision to not buy in the first period and the updated belief about the buyer's valuation is $\pi(\mu, nb)$. In the second period, any optimal information policy must necessarily be a convex combination of the following experiments : $(0, \mu_1^*)$ (μ_1^*, μ_2^*) or $(0, \mu_2^*)$. Now notice buyer v_2 values any such information structure as good as getting no further information. This is because the support of the distribution over posteriors is a subset of $[0, \mu_2^*]$ a region where v_2's optimal one shot decision is to not buy. Given $\mu < \mu_2^*$, he strictly prefers to not buy in the first period. Note further that it is also not optimal for the seller to provide a posterior above μ_2^* with positive probability in the first period. Suppose not i.e the seller chooses to provide $\mu' > \mu_2^*$ with positive probability. By the martingale property, it must be the case that a posterior $\mu'' < \mu_0 < \mu_2^*$ is provided with positive probability as well. Now clearly, it must be the case that $V^2(\mu', \pi_0) = 2p$ and since v_2 does not buy at $\mu'' < \mu_2^*$, we have $V^2(\mu', \pi_0) < 2p$. Now the seller could shift some weight from μ'' and μ' to a posterior $\mu_2^* < \mu' - \epsilon < \mu'$, preserving the martingale property but obtaining a strictly higher payoff since $V^2(\mu' - \epsilon, \pi_0) = 2p$. Hence, in any equilibrium, the seller keeps posteriors in the region $[0, \mu_2^*]$. Finally, we shall focus on seller-optimal equilibrium i.e for any posterior μ, $V^2(\mu, \pi_0)$ is the highest payoff attained by the seller in any equilibrium. These observations allow us

Proposition 1. *If $\pi_0 < 1 - \sigma$, in any seller-optimal equilibrium, the value function V^2 of the seller is :*

$$V^2(\mu; \pi_0) = \begin{cases} 2p & \mu \geq \mu_2^* \\ \pi_0(2p) + (1 - \pi_0)\dfrac{\mu}{\mu_2^*}(p) & \mu \in [H^2, \mu_2^*) \\ \dfrac{\mu}{\mu_2^*}(p) & \mu < H^2 \end{cases} \tag{2}$$

where $H^2 := H(\mu_1^*, \mu_2^*) = \dfrac{2}{\dfrac{1}{\mu_1^*} + \dfrac{1}{\mu_2^*}}$ denotes the harmonic mean of μ_1^* and μ_2^*. The optimal information policy is to choose $(0, \mu_2^*)$ in the first period and subsequently not provide any information if the good is not bought.

Proof. From the observation made above, we already know that if $\mu < \mu_2^*$, in any equilibrium, the buyer of type v_2 would not buy in the first period. Hence, we need only check which actions for type v_1 are sustainable in equilibrium in the first period. We now derive the value function by taking cases for different values of μ. Consider first the case $\mu < H^2$.

We argue in this case there cannot be any equilibrium where type v_1 chooses to buy with strictly positive probability. Suppose not i.e suppose there is an equilibrium where type v_1 chooses to not buy with probability $\alpha < 1$. In this equilibrium, it must be the case that $\pi(\mu, nb) < \pi_0 < 1 - \sigma = \frac{\mu_1^*}{\mu_2^*}$. Hence, following (μ, nb), the seller would optimally choose $(0, \mu_2^*)$. Now consider buyer v_1's incentive to buy at μ. Decisions to buy and not buy yield the following payoffs :

$$\begin{aligned} b &: 2(\mu v_1 - p) \\ nb &: \frac{\mu}{\mu_2^*}(\mu_2^* v_1 - p) \end{aligned} \tag{3}$$

Now a decision to Buy is optimal if and only if :

$$\begin{aligned} 2(\mu v_1 - p) \geq \frac{\mu}{\mu_2^*}(\mu_2^* v_1 - p) \quad &\iff \quad 2(\mu v_1 - p) \geq (\mu v_1 - \frac{p\mu}{\mu_2^*}) \\ &\iff \quad \mu v_1 + \frac{p\mu}{\mu_2^*} \geq 2p \\ &\iff \quad \mu\left(\frac{v_1}{p}\right) + \frac{\mu}{\mu_2^*} \geq 2 \\ &\iff \quad \mu \geq \frac{2}{\frac{1}{\mu_1^*} + \frac{1}{\mu_2^*}} \end{aligned} \tag{4}$$

Since we have $\mu < H^2$, not buying is strictly optimal. Hence we've established that $\alpha < 1$ is not possible in equilibrium. The only possibility left is that type v_1 plays nb. This can indeed be sustained in an equilibrium. The belief following (μ, nb) will be $\pi(\mu, nb) = \pi_0$ and since b will never be played by type v_2, we have $\pi(\mu, b) = 1$. The information following (μ, nb) would be $(0, \mu_2^*)$. Hence, the unique equilibrium involves both buyer types pooling on nb in the first period and yields an expected payoff of $V^2(\mu, \pi_0) := \frac{\mu}{\mu_2^*}(p)$ to the seller.

If $\mu = H^2$, there is a continuum of equilibria. For any $\alpha \in [0, 1]$ one can sustain the type v_1 not buying with probability α in equilibrium (since the inequality in (4) in this becomes an equality). This can be achieved by the above described beliefs and information policy. The equilibrium best for the seller however involves seperation of types and gives a payoff of $V^2(\mu, \pi_0) = \pi_0(2p) + (1 - \pi_0)\frac{\mu}{\mu_2^*}(p)$. Now for the case $H^2 < \mu < \mu_2^*$, we get that the unique equilibrium involves seperation with type v_1 now choosing to buy. The information following (μ, nb) is $(0, \mu_2^*)$. This equilibrium again gives the seller a payoff of $\pi_0(2p) + (1 - \pi_0)\frac{\mu}{\mu_2^*}(p)$.

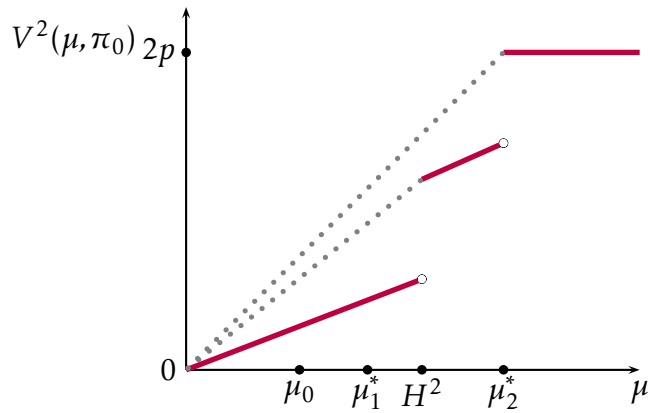
Finally if $\mu_2^* \leq \mu$, then the both buyer types would buy and the seller would provide no further information. Hence, the seller gets a payoff of $2p$. Putting together all of these cases yields exactly the value function described in (2).

We now show the second part of the claim. The value function described is a right continuous non-dereasing piecewise affine function. We have $\lim_{\mu \uparrow H^2} V(\mu, \pi) = \dfrac{H^2 p}{\mu_2^*} < \pi_0(2p) + (1-\pi_0)\dfrac{H^2 p}{\mu_2^*} = V^2(H^2, \pi_0)$. This implies that either the experiment $(0, H^2)$ or the experiment $(0, \mu_2^*)$ is optimal at μ_0 (see figure below). Now, experiment $(0, \mu_2^*)$ is strictly better than the experiment $(0, H^2)$ if and only if

$$
\begin{aligned}
\pi_0 + (1-\pi_0)\frac{H^2}{2\mu_2^*} < \frac{H^2}{\mu_2^*} &\iff \pi_0\Big[1 - \frac{H^2}{2\mu_2^*}\Big] < \frac{H^2}{2\mu_2^*} \\
&\iff \pi_0 < \frac{\frac{H^2}{2\mu_2^*}}{1 - \frac{H^2}{2\mu_2^*}} \\
&\iff \pi_0 < \frac{1}{\frac{2\mu_2^*}{H^2} - 1} \\
&\iff \pi_0 < \frac{1}{\frac{2\mu_2^*}{\frac{2}{\frac{1}{\mu_1^*} + \frac{1}{\mu_2^*}}} - 1} \\
&\iff \pi_0 < \frac{1}{\frac{\mu_1^* + \mu_2^*}{\mu_1^*} - 1} \\
&\iff \pi_0 < \frac{\mu_1^*}{\mu_2^*}
\end{aligned}
$$

Hence, by hypothesis $\pi_0 < \dfrac{\mu_1^*}{\mu_2^*} = 1 - \sigma$, we have that $(0, \mu_2^*)$ is strictly preferred and is hence the optimal experiment for the seller in the first period. $\square$

The following diagram demonstrates the value function for the case $\pi_0 < 1 - \sigma$:



Let us now intepret what is conveyed by Proposition 1. If the seller provides the experiment $(0,\mu_2^*)$ and subsequently the posterior μ_2^* is realised, both buyer types are sufficiently convinced that the match quality is good and would wish to buy. This leads to the best payoff $(2p)$ the seller could wish to achieve. Now compare this policy to providing the experiment $(0,H^2)$. If the posterior is H^2, then seller is able to screen the buyer for his type. The buyer type v_1 would buy and the buyer type v_2 not. The above result essentially states that this screening effort is less beneficial compared to the attempt to convince both buyers to buy by providing the posterior μ_2^*. Note further that this is not trivially apparent since obtaining a payoff $2p$ involves providing the posterior μ_2^* with probability $\frac{\mu_0}{\mu_2^*}$. Now $\frac{\mu_0}{\mu_2^*} < \frac{\mu_0}{H^2}$ where the latter is the probability with which posterior H^2 is provided leading to a payoff lower that $2q$. Hence, the precise trade-off is obtaining a higher payoff with a lower probability versus obtaining a lower payoff with a higher probability. We now consider the case when $\pi_0 \geq \frac{\mu_1^*}{\mu_2^*}$ and derive the seller-optimal value function. This is achieved by the establishing the following proposition.

Proposition 2. *If $\pi_0 \geq 1-\sigma$, then the seller-optimal value function is :*

$$V^2(\mu;\pi_0) = \begin{cases} 2p & \mu \geq \mu_2^* \\ \pi_0(2p) + (1-\pi_0)\dfrac{\mu}{2\mu_2^*}(p) & \mu \in [H^2,\mu_2^*) \\ \pi_0(1-\alpha^*)(2p) + [(1-\pi_0)+\pi_0\alpha^*]\dfrac{\mu}{\mu_2^*}(p) & \mu \in [\mu_1^*,H^2) \\ \dfrac{\pi_0\mu}{\mu_1^*}(p) & \mu < \mu_1^* \end{cases} \tag{5}$$

where $\alpha^* = \left(\frac{1-\pi_0}{\pi_0}\right)\left(\frac{\mu_1^*}{\mu_2^*-\mu_1^*}\right)$ and H^2 denotes the harmonic mean of μ_1^* and μ_2^* as before. Further, if $1-\sigma \leq \pi_0 < 1-\sigma^2$, then it is optimal for the seller to attempt to screen in the first period by providing the posterior H^2. If $\pi_0 \geq 1-\sigma^2$, then it is optimal for the seller to attempt to partially screen i.e the buyer type v_1 buys with probability $1-\alpha^*$ in the first period. At

$\pi_0 = 1 - \sigma^2$, *the seller is indifferent between screening and partial screening.*

Proof. As in the proof of the previous proposition, we derive the value function by taking cases. Consider first the case when $\mu < \mu_1^*$. We argue as before that in this case the unique equilibrium that follows μ involves both buyer types pool on nb. Notice first that at $\mu < \mu_1^* < \mu_2^*$ if no further information about the match quality is provided, both buyer types strictly prefer to not buy. This in turn implies that both buyer types strictly prefer to not buy at μ if at all any information is provided following (μ, nb) since obtaining any amount of information is atleast at good at getting no further information. The seller obtains a payoff $V^2(\mu, \pi_0) = \dfrac{\pi_0 \mu}{\mu_1^*}(q)$.

Now consider the case $\mu = \mu_1^*$. There is now a continuum of equilibria with the buyer v_1 playing nb with probability $\alpha \in [(\frac{1-\pi_0}{\pi_0})(\frac{\mu_1^*}{\mu_2^*-\mu_1^*}), 1]$. We first show that $\alpha < (\frac{1-\pi_0}{\pi_0})(\frac{\mu_1^*}{\mu_2^*-\mu_1^*})$ cannot be supported. Suppose not. If indeed an equilibrium can be supported with such an α, then the beliefs of the seller must be such that $\pi(\mu, nb) < \dfrac{\mu_1^*}{\mu_2^*}$. Hence the information after (μ, nb) must be $(0, \mu_2^*)$. Once again it follows that type v_1 would weakly prefer to play b if and only if $\mu \geq H^2$ which is a contradiction. Now consider any $\alpha \in [(\frac{1-p}{p})(\frac{\mu_1^*}{\mu_2^*-\mu_{t_1}^*}), 1]$. We show that an equilbrium can be sustained with any such α. Following (μ, nb), the belief would be $\pi(\mu, nb) \geq \pi_0$. Hence the information structure (μ_1^*, μ_1^*) would be optimal for the seller. Hence, for the type v_1 buyer, actions b and nb would result in the same payoff and he would hence be indifferent between the two actions. The payoff to the seller in equilibrium would be $\pi_0(1-\alpha)(2p) + [(1-\pi_0) + \pi_0\alpha](p)$ which is strictly decreasing in α and maximised over $[(\frac{1-\pi_0}{\pi_0})(\frac{\mu_1^*}{\mu_2^*-\mu_1^*}), 1]$ at the value $\alpha^* = (\frac{1-\pi_0}{\pi_0})(\frac{\mu_1^*}{\mu_2^*-\mu_1^*})$. Now using $\mu = \mu_1^*$ and the expression for α^*, the seller's payoff would be $V^2(\mu, \pi_0) = \pi_0(1-\alpha^*)(2p) + [(1-\pi_0) + \pi_0\alpha^*]\dfrac{\mu}{\mu_2^*}(p)$.

Now suppose $\mu_1^* < \mu < H^2$ and consider first the case $\pi_0 > \dfrac{\mu_1^*}{\mu_2^*}$. We argue that the unique equilbrium involves type v_1 playing nb with probability α^*. We first argue there can be no pooling on nb. If it were possible then the belief following (μ, nb) would be $\pi(\mu, nb) = \pi_0$ and information (μ_1^*, μ_2^*) would be provided. Now notice that this information structure is as valuable for v_1 as getting no further information. Since $\mu_1^* < \mu$, type v_1 would strictly prefer to buy. There cannot be a seperating equilibrium either. If seperation were sustainable then following (μ, nb), information $(0, \mu_2^*)$ would be provided. For type v_1 to prefer a_0 to a_1, it must be the case that $\mu \geq H^2$ which is a contradiction. Finally, we can show that the unique equilibrium outcome in this case involves the buyer type v_1 playing nb with probability α^* and following (μ, nb) the belief of the seller is $\pi(\mu, nb) = \dfrac{\mu_1^*}{\mu_2^*}$ and ran-

domises between information structures (μ_1^*, μ_2^*) and $(0, \mu_2^*)$ with probability $z^* = \frac{\frac{\mu}{\mu_1^*} - 1}{1 - \frac{\mu}{\mu_2^*}}$ on information structure $(0, \mu_{t_2}^*)$ to keep the buyer indifferent. The payoff to the seller is again $V^2(\mu, \pi_0) = \pi_0(1-\alpha^*)(2p) + [(1-\pi_0) + \pi_0\alpha^*]\frac{\mu}{\mu_2^*}(p)$. One can argue that this is also the seller optimal value when $\pi_0 = \frac{\mu_1^*}{\mu_2^*}$.

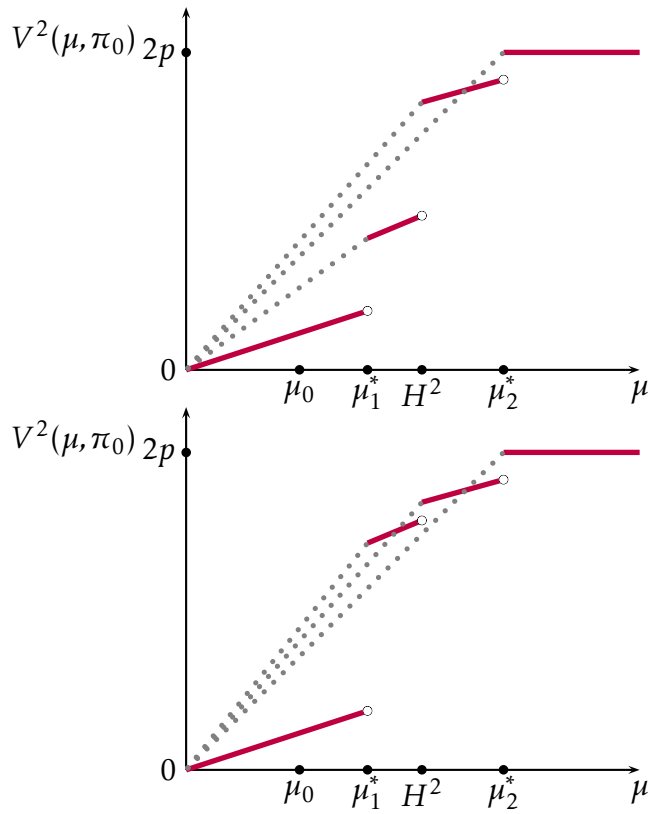
If $H^2 \leq \mu < \mu_2^*$, we can show that separation can be supported as an equilibrium and yields the best payoff to the seller equal to $V^2(\mu, \pi_0) = \pi_0(2p) + (1-\pi_0)\frac{\mu}{2\mu_2^*}(p)$. Finally if $\mu_2^* \geq \mu$, then there is an equilibrium where the seller gives no further information and gets a payoff of $V^2(\mu, \pi_0) = 2q$. The culmination of all of these cases yields the value function in (5).

Note now that the value function is again a non-decreasing left continuous piecewise linear function. The concavification of this value function yields the payoff obtained from the optimal signal to the seller. Note that $\lim_{\mu \uparrow \mu_1^*} V^2(\mu; \pi_0) = \frac{\pi_0 \mu}{\mu_1^*}(p) < \pi_0 q[2-\alpha^*] = V^2(\mu_1^*; \pi_0)$ and $\lim_{\mu \uparrow H^2} V^2(\mu; \pi_0) = \pi_0(1-\alpha^*)(2p) + [(1-\pi_0) + \pi_0\alpha^*]\frac{H^2}{\mu_2^*}(p) < \pi_0(2p) + (1-\pi_0)\frac{H^2}{\mu_2^*}(p) = V^2(H^2; \pi_0)$. Using the concavification of V^2, we can establish it is optimal for the seller to partially screen if and only if

$$\begin{aligned}
\pi_0[2-\alpha^*] &\geq \frac{\mu_1^*}{H^2}\left[2\pi_0 + (1-\pi_0)\frac{H^2}{\mu_2^*}\right] \\
&= \frac{2\pi_0\mu_1^*}{H^2} + (1-\pi_0)\frac{\mu_1^*}{\mu_2^*} \\
&= \frac{2\pi_0\mu_1^*}{\frac{2\mu_1^*\mu_2^*}{\mu_1^* + \mu_{t_2}^*}} + (1-\pi_0)\frac{\mu_1^*}{\mu_2^*} \\
&= \frac{\mu_1^*}{\mu_2^*} + \pi_0
\end{aligned}$$

which happens if and only if $\pi_0(1-\alpha^*) \geq \frac{\mu_1^*}{\mu_2^*}$. This is equivalent to $\pi_0 \geq \frac{\mu_1^*}{\mu_2^*}\left(\frac{2\mu_2^* - \mu_1^*}{\mu_2^*}\right) = 1 - \sigma^2$ and the conclusion follows. □

The following diagram illustrates the value function derived in the above proposition. The first diagram corresponds to the case when $1-\sigma \leq \pi_0 < 1-\sigma^2$ and the second diagram corresponds to the case $1-\sigma^2 < \pi_0 < 1$:



Let us now discuss and interpret the points conveyed by the above proposition. The closed-form of the value function derived applies for any $\pi_0 > 1-\sigma$. However, the shape of the value function and consequently the choice of optimal experiment depends crucially on the *level* of π_0. If the level is intermediate i.e $1-\sigma \leq \pi_0 < 1-\sigma^2$, then the optimal experiment is $(0, H^2)$. If the posterior H^2 is realised, seperation of types takes place with the high valuation buyer v_1 buying and the low valuation buyer v_2 not buying. Hence, an attempt to screen for the buyer's type is optimal for the seller.

If the level is higher i.e $\pi_0 > 1-\sigma^2$, the seller chooses the experiment $(0, \mu_1^*)$ in the first period. Now subsequently, if μ_1^* is realised, the high valuation buyer randomises between buying and not buying. If a decision to not buy is observed by the seller, no further information is provided and buyer v_1 buys in the next period. Since the belief μ_1^* reached is exactly the high valuation buyer's threshold, he is indifferent between buying today and not buying. This essentially corresponds to a situation where the belief about the high valuation buyer is high so that the seller finds it beneficial to focus only on persuading him to buy. He does so by attempting to push beliefs only upto his threshold wherein v_1 either buys with some probability in the first period or defers the decision to buy to the next period.

Let us also try to interpret what Propositions 1 and 2 imply together. An important feature that is conveyed is a *monotonicity* of the information provided in the first period with respect to the prior belief π_0 about the buyer type v_1. The choice of experiment in the first period depends on the level of π_0. In particular, there are 3 regions for π_0. Namely, $\left[0, 1-\sigma\right)$, $\left[1-\sigma, 1-\sigma^2\right)$ and $\left[1-\sigma^2, 1\right]$. The experiments corresponding to the 3 regions are $(0, \mu_2^*)$, $(0, H^2)$ and $(0, \mu_1^*)$ respectively. Note they are decreasing in the level of informativeness. Hence, the higher the value of π_0, the lesser the informativeness of the experiment chosen in the first period. This is intuitively plausible. Consider first the extreme cases. If types were known perfectly, the high valuation buyer would obtain $((0, \mu_1^*))$ less information than a low valuation buyer $((0, \mu_2^*))$[8]. Hence, if the seller has higher beliefs about the high valuation buyer, he would provide less information. This is reflected in the monotonicity conveyed by the above results.

4 T-period Interaction

We now consider the game with T periods of interaction. As discussed before, the seller's incentives involve framing information in a way that convinces the buyer that the match quality is indeed likely to be good. The buyer's decisions would involve trading off current expected payoffs from purchasing the durable for information he could potentially acquire by deferring his buying option. Additionally, as in the two-period game studied above, we shall observe that the seller's information policy would satisfy monotonicity in his belief about the buyer's valuation. The more likely he considers the case that it is the high valuation buyer, the less is the information he provides about match quality. In particular, we shall again see that his strategy will depend on the level of this belief. For the lowest level (low valuation buyer with high probability) he would attempt to persuade both buyer types simultaneously. For the highest level (high valuation buyer with high probability), he would wish to focus only on persuading the high valuation buyer. Along with these features, we shall derive various other qualitative features that would emerge in equilibrium.

We now formally begin the analysis by defining the game and strategies. Recall that our equilibrium concept will be Perfect Bayesian Equilibrium (PBE). We denote as $\hat{G}^T(\mu, \pi)$ the T-period game when the beliefs about the match quality and buyer type are (μ, π).

[8]When we say "less" information and compare experiments, we are using the *Blackwell informativeness ranking*. See Blackwell (1951)[5]. Since $(0, \mu_2^*)$ is a mean preserving spread of $(0, \mu_1^*)$, it is more informative in the Blackwell sense.

Note that in $\hat{G}^T(\mu,\pi)$, the seller moves first. It shall be useful to define $G^T(\mu,\pi)$ to be the game where the buyer moves first and beliefs are (μ,π). Note that the game $\hat{G}^T(\mu,\pi)$ simply comprises of the seller's choice of an experiment followed by the game $G^T(\mu',\pi)$ where μ' is the posterior generated by the experiment chosen. We now define strategies for the T-period game. Since the buyer is privately informed he only holds beliefs about the state of the world at every private history. On the other hand, the seller holds beliefs both about the match quality and the buyer's type. In the section that follows, we shall focus attention to a particular class of strategies namely *Markovian Strategies* and then construct equilibria within the class.

4.0.1 Markovian Strategies

We focus our attention on strategies (for both the seller and the buyer) which specify actions as a function over the current belief about ω; the buyer type i.e the pair (μ,π); and the number of periods left t (including the current stage) in the game. Hence, actions taken by agents depend on past play insofar as they affect the *state variable* of the game (μ,π,t). We formally define the class of strategies as follows :

Definition 3. *A Markovian Strategy for the seller is a map* $\sigma_S : [0,1]^2 \times \{1,2,...,T\} \to \Delta(\mathcal{E})$ *and for the buyer* v *is a map* $\sigma_v : [0,1]^2 \times \{1,2,...,T\} \to \Delta(A)$.

Hence, when the current belief pair is (μ,π) and t periods are left including the current stage, the agents choose actions $\sigma_S(\mu,\pi,t)$ and $\sigma_v(\mu,\pi,t)$ respectively. Note that this does not imply that agents choose actions at the *same* (μ,π,t). When the seller chooses an experiment $e \in \mathcal{E}$ at (μ,π,t), it generates a posterior μ' and subsequently the buyer chooses an action contingent on (μ',π,t).

In order to interpret Markovian strategies, we urge the reader to view (μ,π,t) as a *state variable* associated with the game. Player actions hence depend on histories in so far as they influence the state variable (μ,π,t). Two remarks are now in order :

Remark 1. *(Off-path beliefs) : Note that any specification of Markovian strategies completely describes on-path play in the game. In order to specify behaviour off-path via Markovian strategies, one would additionally need to describe off-path beliefs. Hence, any Markovian strategy profile* $< \sigma_S, \sigma_{v_1}, \sigma_{v_2} >$ *in the sense of Definition 3 coupled with a system of beliefs* $< \pi_S, \pi_{v_1}, \pi_{v_2} >$ *from Definition 2 would induce a strategy profile* $< \sigma_S, \sigma_{v_1}, \sigma_{v_1} >$ *in the sense of Definition 1.*

Remark 2. *(Continuation values) : Given a Markovian strategy profile, we can define continuation values for each player as a function of the triple (μ, π, t). This is summarised in the following table :*

.	$\hat{G}^t(\mu,\pi)$	$G^t(\mu,\pi)$
S	$\hat{V}_S^t(\mu,\pi)$	$V_S^t(\mu,\pi)$
v_1	$\hat{V}_1^t(\mu,\pi)$	$V_1^t(\mu,\pi)$
v_2	$\hat{V}_2^t(\mu,\pi)$	$V_2^t(\mu,\pi)$

Hence, $\hat{V}_S^t(\mu,\pi), \hat{V}_1^t(\mu,\pi), \hat{V}_2^t(\mu,\pi)$ denote continuation values at (μ,π,t) and the game is $\hat{G}^t(\mu,\pi)$ i.e before the seller moves. Similarly, $V_S^t(\mu,\pi), V_1^t(\mu,\pi), V_2^t(\mu,\pi)$ denotes continuation value at (μ,π,t) in $G^t(\mu,\pi)$ after the seller moves and before the buyer moves.

Consider Remark 2 and the seller's incentives at (μ,π,t) in the game $\hat{G}^t(\mu,\pi)$. Since the belief about ω is μ, any choice of experiment $e \in \mathcal{E}$ would result in a distribution over posteriors (with countable support) $\tau_e \in \Delta(\Delta(\Omega))$ such that the martingale property is satisfied $\mathbb{E}_{\tau_e}[\mu'] = \mu$. Define the set $\Gamma(\mu) = \{\tau \in \Delta(\Delta(\Omega)) : \mathbb{E}_\tau[\mu'] = \mu \text{ and } \text{supp}(\tau) \text{ is countable}\}$ i.e the set of all such distributions. It is well known[9] that there is a one-to-one correspondence between this set and the set $\mathcal{E}$ i.e all experiments with a countable message space M. Since the continuation values $V_S^t(\mu',\pi)$ for the seller depend only the posteriors μ' generated by $e \in \mathcal{E}$, we get the following relation :

$$\hat{V}_S^t(\mu,\pi) = \sup_{\tau \in \Gamma(\mu)} \sum_{\mu' \in \text{supp}(\tau)} V_S^t(\mu',\pi)\tau(\mu') \tag{6}$$

It is further known from Aumman-Maschler (1966)[2] and Kamenica-Gentzkow (2011)[17] that the above relation is in turn equivalent to :

$$\hat{V}_S^t(\mu,\pi) = \text{cav}[V_S^t(\mu,\pi)] \tag{7}$$

where cav denotes the concavification operator[10]. These two relations are very useful in understanding the incentives of the seller : choice of experiment and the continuation value that accrues from it.

Now let us consider the incentives of buyer type $v_i \in \mathcal{V}$. Suppose along a particular history, the game reaches the point (μ,π,t) and the action nb shifts the belief on $\mathcal{V}$ from π to

[9] See for eg. Blackwell (1951) [5] and Blackwell (1953) [6]

[10] The concavification of a bounded function, $f : [0,1] \to \mathbb{R}$ is the pointwise infimum of all concave functions which pointwise dominate f. See appendix for more details.

π'. The expected long run payoffs from decisions to buy and not buy are as follows :

$$b \;:\; (\mu v_i - p)t \tag{8}$$

$$nb \;:\; 0 + \hat{V}_i^{t-1}(\mu, \pi') \tag{9}$$

Hence, if buyer v_i is indifferent, the above two values must be equal. Before we proceed towards construction of strategies, it shall be useful to establish the following result :

Claim 1. *In any equilibrium, at any stage :*

1. *If $\mu < \mu_i^*$, then buyer v_i strictly prefers to not buy*
2. *If $\mu < \mu_2^*$ and $\pi < 1$, seller does not provide any posterior $\mu' > \mu_2^*$ with positive probability*
3. *If $\mu < \mu_2^*$ and $\pi = 1$, it is optimal for the seller to keep posteriors less than or equal to μ_2^**

Proof. 1. Suppose $\mu < \mu_i^*$ and t periods are left. If buyer v_i chooses to buy, then he obtains an expected long run payoff equal to $(\mu v_i - p)t$. If he chooses to not buy, he obtains an expected long run payoff equal to $\hat{V}_i^{t-1}(\mu, \pi')$. Consider now the situation where no further information about ω is provided for the rest of the game if the buyer chooses to not buy. Since $\mu < \mu_i^*$, the maximum expected payoff the buyer achieves from not buying equals 0 (follows from a straighforward induction argument). Now further note that $\mu < \mu_i^*$ implies that $(\mu v_i - p)t < 0$. But since getting any amount of information is weakly better than getting no further information, we have that $\hat{V}_i^{t-1}(\mu, \pi') \geq 0$. Hence, $(\mu v_i - p)t < \hat{V}_i^{t-1}(\mu, \pi')$ implying that buyer v_i strictly prefers to not buy.

2. Suppose the game is at a stage (μ, π, t) where $\mu < \mu_2^*$ and $\pi < 1$ and the seller chooses an experiment $\tau \in \Gamma(\mu)$ such that there exists $\mu' > \mu_2^*$ with $\tau(\mu') > 0$. We will show that this is not possible in equilibrium in that the seller has a profitable deviation. Observe first that for any $\mu'' > \mu_2^*$ it must be the case that $V_S^t(\mu'', \pi) = pt$ since the seller can assure himself a payoff of pt by witholding information for the remainder of the game (because $\mu'' > \mu_2^* > \mu_1^*$ induces both buyer types to buy immediately). Consider the set $\mathcal{M}(\mu') := \text{supp}(\tau) \backslash \{\mu'\}$. Now we must have that $V^t(\mu'', \pi) < pt$ for some $\mu'' \in \mathcal{M}(\mu')$. This follows from the martingale property satisfied by τ and part 1) above. From the martingale property, since $\mu' > \mu_2^* > \mu$ is provided with positive probability, there also exists a posterior $\mu'' < \mu$ such that $\tau(\mu'') > 0$. This means $\mu'' < \mu_2^*$ and hence buyer v_2 strictly prefers to not buy at (μ'', π, t). Since we have $\pi < 1$, it follows that $V^t(\mu'', \pi) < pt$.

Now, let $\mu'' \in \mathcal{M}(\mu')$ be such that $V^t(\mu'', \pi) < pt$ and define $\bar{m} := \tau(\mu') + \tau(\mu'')$ and let $\epsilon > 0$ be such that $0 < \frac{\mu' - \mu''}{\mu' - \mu'' - \epsilon}\tau(\mu') < \bar{m}$ and $\mu_2^* < \mu' - \epsilon < \mu'$. Now additionally define $\mu'_\epsilon := \mu' - \epsilon$ and the following experiment $(\hat{\tau})$:

$$\hat{\tau}(\nu) = \begin{cases} \tau(\nu) & \text{if } \nu \in \mathcal{M}(\mu')\backslash\{\mu''\} \\ \dfrac{\mu' - \mu''}{\mu' - \mu'' - \epsilon}\tau(\mu') & \text{if } \nu = \mu'_\epsilon \\ \bar{m} - \dfrac{\mu' - \mu''}{\mu' - \mu'' - \epsilon}\tau(\mu') & \text{if } \nu = \mu'' \end{cases}$$

It is straightforward to verify that $\hat{\tau}$ is indeed an experiment centered at μ. Notice also that $\tau(\mu') + \tau(\mu'') = \bar{m} = \hat{\tau}(\mu'_\epsilon) + \hat{\tau}(\mu'')$ and that $\hat{\tau}(\mu'_\epsilon) > \tau(\mu')$ and $\hat{\tau}(\mu'') < \tau(\mu'')$. These facts together imply that

$$\sum_{\nu \in \text{supp}(\hat{\tau})} V^t(\nu, \pi)\hat{\tau}(\nu) > \sum_{\nu \in \text{supp}(\tau)} V^t(\nu, \pi)\tau(\nu)$$

as a result, the seller can profitably deviate to $\hat{\tau}$.

3. Now consider the case when $\pi = 1$. This implies that the seller believes with probability one that it is buyer v_1. Now suppose it is the case that at (μ, π, t), the seller chooses experiment τ and there exists $\mu' > \mu_2^*$ such that $\mu' \in \text{supp}(\tau)$. This means that $V^t(\mu', \pi) = pt$. Now consider the set $\mathcal{M}^+ := \{\mu' \in \text{supp}(\tau) : \mu' > \mu_2^*\}$ and define the experiment $\hat{\tau}$ as follows :

$$\hat{\tau}(\nu) = \begin{cases} \tau(\nu) & \text{if supp}(\tau)\backslash\mathcal{M}^+ \\ \displaystyle\sum_{\nu' \in \mathcal{M}^+} \tau(\nu') & \text{if } \nu = \mu_2^* \end{cases}$$

Clearly, we have that $\mathbb{E}_\tau[V^t(\nu, \pi)] = \mathbb{E}_{\hat{\tau}}[V^t(\nu, \pi)]$. Hence the experiment $\hat{\tau}$ gives the same payoff as τ. Note further that $\text{supp}(\hat{\tau}) \subseteq [0, \mu_2^*]$. Hence, the desired result follows.

□

4.0.2 Construction of Strategies

In this section we provide an explicit construction and closed-form characterisation of an equilibrium in Markovian strategies. In order to proceed, it shall be useful to estab-

lish certain expressions, values and notation. These shall be crucial in defining various equilibrium objects.

1. (*Cut-offs for μ*) : For any $0 \leq l \leq t$, where $l, t \in \mathbb{N}$, define

$$\bar{\mu}_l^t := \frac{t}{l\dfrac{1}{\mu_1^*} + (t-l)\dfrac{1}{\mu_2^*}}$$

   Note that $\bar{\mu}_l^t$ is the weighted harmonic mean of the buyer thresholds μ_1^* and μ_2^* with a weight of l on μ_1^* and a weight of $(t-l)$ on μ_2^*. Note further that $0 \leq l < k \leq t$ implies $\bar{\mu}_k^t < \bar{\mu}_l^t$; $0 \leq l \leq t < t'$ implies $\bar{\mu}_l^t < \bar{\mu}_l^{t'}$; for any $l \geq 0$, $\lim_{t\to\infty} \bar{\mu}_l^t = \mu_2^*$.

2. (*Cut-offs for π*) : For any $l \in \mathbb{N}$, define

$$\bar{\pi}_l := 1 - \sigma^l$$

   where $\sigma = 1 - \dfrac{\mu_1^*}{\mu_2^*}$ is the threshold gap. Note that since $\sigma \in (0,1)$, we have that $l < l'$ implies $\bar{\pi}_l < \bar{\pi}_{l'}$ i.e. the sequence $\{\bar{\pi}_k\}_{k\in\mathbb{N}}$ is strictly increasing in k. Moreover, observe that $\lim_{l\to\infty} \bar{\pi}_l = 1$.

3. (*Shift of beliefs on buyer's valuation*) : For any values $0 \leq \pi' \leq \pi \leq 1$, $\alpha \in [0,1]$ we write $\pi \longrightarrow_\alpha \pi'$ if it is true that $\dfrac{\pi\alpha}{\pi\alpha + (1-\pi)} = \pi'$. The interpretation is that α is the probability with which buyer v_1 chooses to not buy which shifts the belief on v downward from π to π' given that buyer v_2 chooses to not buy for sure.

4. (*Denoting Experiments*) : Lastly, as in the analysis of the two-period game, we shall represent experiments with support containing atmost two points as follows : For any belief μ about match quality and μ', μ'' such that $\mu' \leq \mu \leq \mu''$, denote (μ', μ'') as the experiment which provide posterior μ' with probability q and posterior μ'' with probability $1-q$ such that $q\mu' + (1-q)\mu' = \mu$. Note that under this equality, the resulting distribution over posteriors is uniquely determined and hence the notation (μ', μ'') conveys it precisely.

The strategies of the agents are as follows :

Buyer v_2's Strategy : At every period, buyer v_2 plays according to the same cut-off strategy : Choose to buy if and only if $\mu \geq \mu_2^*$. We formally describe the strategies below :

$$\sigma_{v_2}^t(\mu, \pi) = \begin{cases} b & \text{if } \mu \geq \mu_2^* \\ nb & \text{if } \mu < \mu_2^* \end{cases}$$

for all $(\mu, \pi, t) \in [0,1]^2 \times \{1, ..., T\}$

Buyer v_1's Strategy : Fix $0 < t \leq T$. The strategy of buyer v_1 is described as follows :

1. *Case 1* : Suppose $\pi = 1$

$$\sigma^t_{v_1}(\mu, 1) = \begin{cases} b & \text{if } \mu \geq \mu^*_1 \\ nb & \text{if } \mu < \mu^*_1 \end{cases}$$

2. *Case 2* : Let $0 \leq l < t$ and $\pi \in [\bar{\pi}_l, \bar{\pi}_{l+1})$

$$\sigma^t_{v_1}(\mu, \pi) = \begin{cases} b & \text{if } \mu \geq \bar{\mu}^t_1 \\ \alpha_{l-i} & \text{if } \mu \in [\bar{\mu}^t_{l+1-i}, \bar{\mu}^t_{l-i}) \\ nb & \text{if } \mu < \bar{\mu}^t_{l+1} \end{cases}$$

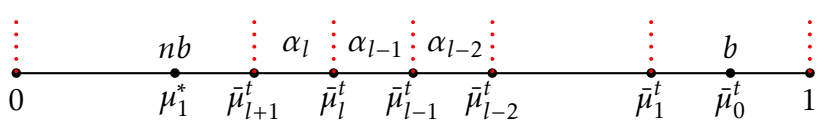
   where $0 \leq i \leq l-1$ and $\pi \longrightarrow_{\alpha_{l-i}} \bar{\pi}_{l-i}$

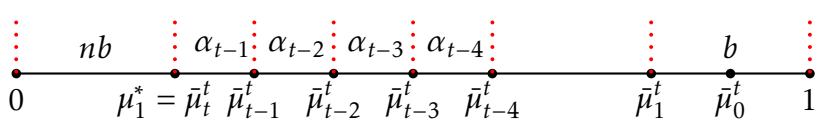
3. *Case 3* : Let $l = t$ and $\pi \in [\bar{\pi}_t, 1)$

$$\sigma^t_{v_1}(\mu, \pi) = \begin{cases} b & \text{if } \mu \geq \bar{\mu}^t_1 \\ \alpha_{t-i} & \text{if } \mu \in [\bar{\mu}^t_{t+1-i}, \bar{\mu}^t_{t-i}) \\ nb & \text{if } \mu < \bar{\mu}^t_t \end{cases}$$

   where $1 \leq i \leq t-1$ and $\pi \longrightarrow_{\alpha_{t-i}} \bar{\pi}_{t-i}$

The following diagrams illustrate cases 2 and 3 :





seller's Strategy : Fix $0 < t \leq T$. The strategy of the seller is described as follows :

1. *Case 1* : Let $0 < t$ and $\pi = 1$

$$\sigma^t_S(\mu, 1) = \begin{cases} (\mu, \mu) & \text{if } \mu \geq \mu^*_1 \\ (0, \mu^*_1) & \text{if } \mu < \mu^*_1 \end{cases}$$

2. *Case 2* : Let $0 \leq l < t$ and $\pi \in (\bar{\pi}_l, \bar{\pi}_{l+1})$

$$\sigma_S^t(\mu, \pi) = \begin{cases} (\mu, \mu) & \text{if } \mu \geq \bar{\mu}_0^t \\ (\bar{\mu}_{l-i}^t, \bar{\mu}_{l-i-1}^t) & \text{if } \mu \in [\bar{\mu}_{l-i}^t, \bar{\mu}_{l-i-1}^t) \\ (0, \bar{\mu}_l^t) & \text{if } \mu < \bar{\mu}_l^t \end{cases}$$

where $0 \leq i \leq l-1$

3. *Case 3* : Let $l = t$ and $\pi \in (\bar{\pi}_t, 1)$

$$\sigma_S^t(\mu, \pi) = \begin{cases} (\mu, \mu) & \text{if } \mu \geq \bar{\mu}_0^t \\ (\bar{\mu}_{t-i}^t, \bar{\mu}_{t-i-1}^t) & \text{if } \mu \in [\bar{\mu}_{t-i}^t, \bar{\mu}_{t-i-1}^t) \\ (0, \bar{\mu}_l^t) & \text{if } \mu < \bar{\mu}_t^t \end{cases}$$

where $0 \leq i \leq t-1$

4. *Case 4* : Let $0 \leq l \leq t$ and $\pi = \bar{\pi}_l$

$$\sigma_S^t(\mu, \pi) = \begin{cases} (\mu, \mu) & \text{if } \mu \geq \bar{\mu}_0^t \\ (\bar{\mu}_{l-i}^t, \bar{\mu}_{l-i-1}^t) & \text{if } \mu \in [\bar{\mu}_{l-i}^t, \bar{\mu}_{l-i-1}^t) \\ < z(\mu, l, t) : (0, \bar{\mu}_{l-1}^t); 1 - z(\mu, l, t) : (\bar{\mu}_l^t, \bar{\mu}_{l-1}^t) > & \text{if } \mu \in [\bar{\mu}_l^t, \bar{\mu}_{l-1}^t) \\ < z'(\mu, l, t) : (0, \bar{\mu}_{l-1}^t); 1 - z'(\mu, l, t) : (0, \bar{\mu}_l^t) > & \text{if } \mu < \bar{\mu}_l^t \end{cases}$$

where $1 \leq i \leq l-1$

where $z(\mu, l, t), z'(\mu, l, t)$ are defined as follows :

- *Defining* $z'(\mu, l, t)$: Consider $\mu \in [0, \bar{\mu}_l^t)$. If there exists $\hat{z}' \in [0, 1]$ such that the following equation is satisfied

$$(\mu v_1 - p)(t+1) = \hat{z}' \frac{\mu}{\bar{\mu}_{l-1}^t} (\bar{\mu}_{l-1}^t v_1 - p)t + (1 - \hat{z}') \frac{\mu}{\bar{\mu}_l^t} (\bar{\mu}_l^t v_1 - p)t \tag{10}$$

  then define $z'(\mu, l, t) = \hat{z}'$. If the above equation has no solution $\hat{z}' \in [0, 1]$, then define $z'(\mu, l, t) = 1$. Notice that if a solution $\hat{z}' \in [0, 1]$ exists, it must be unique. Moreover, the set of all μ such that there exists a solution is precisely $[\bar{\mu}_{l+1}^{t+1}, \bar{\mu}_l^t) \subset [0, \bar{\mu}_l^t)$

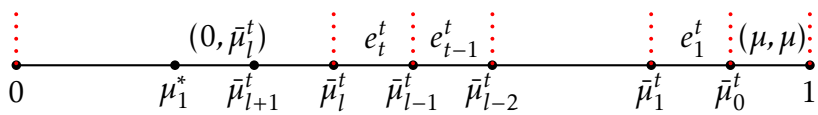
- *Defining* $z(\mu, l, t)$: Consider $\mu \in [\bar{\mu}_l^t, \bar{\mu}_{l-1}^t)$. If there exists $\hat{z} \in [0, 1]$ such that the fol-

lowing equation is satisfied

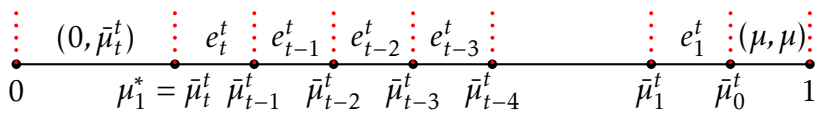
$$(\mu v_1 - p)(t+1) = \hat{z}' \frac{\mu}{\bar{\mu}^t_{l-1}} (\bar{\mu}^t_{l-1} v_1 - p)t + (1-\hat{z}')(\mu v_1 - p)t \tag{11}$$

then define $z(\mu, l, t) = \hat{z}$. If the above equation has no solution $\hat{z} \in [0,1]$, then define $z(\mu, l, t) = 1$. Notice that if a solution $\hat{z} \in [0,1]$ exists, it must be unique. Moreover, the set of all μ such that there exists a solution is precisely $[\bar{\mu}^t_l, \bar{\mu}^{t+1}_l] \subset [\bar{\mu}^t_l, \bar{\mu}^t_{l-1})$

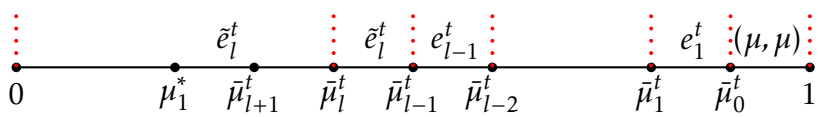
The following diagram illustrates the seller's strategy :



For $\pi \in (\bar{\pi}_t, 1)$



For $\pi = \bar{\pi}_l$ and $0 \le l \le t$



where $\hat{e}^t_l =< z : (0, \bar{\mu}^t_{l-1}); 1-z : (0, \bar{\mu}^t_l) >$, $\tilde{e}^t_l =< z : (0, \bar{\mu}^t_{l-1}); 1-z : (\bar{\mu}^t_l, \bar{\mu}^t_{l-1}) >$ and $e^t_{l-i} = (\bar{\mu}^t_{l-i}, \bar{\mu}^t_{l-i-1})$

Off-path Beliefs : Since actions are perfectly observable, beliefs at any information set are only about match quality and buyer type. Since the buyer is privately informed of his type, his beliefs are only about the match quality but the seller has beliefs about both the state of the world and type.

- seller : $\hat{\pi}_S : \bigcup_{t=0}^{T-1} (\mathcal{E} \times M \times A)^t \to \Delta(\Omega \times \mathcal{V})$
- Buyer (v) : $\hat{\pi}^v_B : \bigcup_{t=0}^{T-1} (\mathcal{E} \times M \times A)^t \times \mathcal{E} \times M \to \Delta(\Omega)$

Suppose we consider the game $\hat{G}^T(\mu_0, \pi_0)$. On path beliefs can be specified by using the strategies above with the initial belief pair (μ_0, π_0) for every history reached with positive probability. If the seller deviates, both players update beliefs about ω according to experiment chosen in the deviation. Now let h be any history reached with zero probability at which the seller is supposed to move. Let μ be the belief about ω induced by the sequence of experiments and their outcomes in h. Further, consider the first time the buyer deviates in the history h. Let π be the belief about the buyer type just one stage before this first deviation takes place (this could be the beginning of the game). We set $\hat{\pi}_S(h) =< \mu, 1-\mu > \times < \pi, 1-\pi >$. Now consider any history h' at which the buyer moves. Note that we can represent this history as $h' = (h, e, m)$ where h is a history at which the seller moves. Let μ be such that the marginal of $\hat{\pi}_S(h)$ is μ. Further let μ' denote the bayesian update of μ using experiment e and its outcome m. We set $\hat{\pi}_{v_i}(h) =< \mu', 1-\mu' >$.

Continuation Values : Recall from Remark 2 in section 4.0.1 that continuation values can be defined once a Markovian strategy profile is specified. We can hence deduce the continuation values for the strategy profile defined above. Recall also that in order to understand the incentives of the seller and his optimal choice of experiments, it is enough to know the function $V_S^t(\mu, \pi)$ (and its concavification). The following result is crucial in examining the structure of $V_S^t(\mu, \pi)$ for the above strategies and implies the optimality of the seller's strategy.

Proposition 3. *Let $V_S^t(\mu, \pi)$ be the value function of the seller in $G^T(\mu, \pi)$ defined by the above strategy profile. For any $0 \leq l < t$ and $\pi \in [\bar{\pi}_l, \bar{\pi}_{l+1})$*

1. *The function $V_S^t(., \pi) : [0,1] \to \mathbb{R}_+$ is non-decreasing, right-continuous and piecewise affine with the following set of break points :*

$$\{\bar{\mu}_{l+1}^t, \bar{\mu}_l^t, ..., \bar{\mu}_1^t, \bar{\mu}_0^t\}$$

2. *Suppose the game is at a stage where (μ, π, t) such that $\mu < \bar{\mu}_l^t$. Let $\succsim_{(\mu,\pi,t)}$ be the seller's preference over experiments in $\Gamma(\mu)$. The following are true :*

   (a) *The experiment $(0, \bar{\mu}_l^t)$ is optimal i.e*

$$(0, \bar{\mu}_l^t) \succsim_{(\mu,\pi,t)} \tau \text{ for all } \tau \in \Gamma(\mu)$$

   (b) *Moreover, the following relationship holds :*

$$(0, \bar{\mu}_l^t) \succsim_{(\mu,\pi,t)} (0, \bar{\mu}_{l-1}^t) \succsim_{(\mu,\pi,t)} ... \succsim_{(\mu,\pi,t)} (0, \bar{\mu}_1^t) \succsim_{(\mu,\pi,t)} (0, \bar{\mu}_0^t)$$

   (c) *If $\pi = \bar{\pi}_l$, then we have :*

$$(0, \bar{\mu}_l^t) \sim_{(\mu,\pi,t)} (0, \bar{\mu}_{l-1}^t).$$

3. *Suppose the game is at a stage where (μ, π, t) such $\mu \in [\bar{\mu}_{l-i}^t, \bar{\mu}_{l-i-1}^t)$ with $0 \leq i \leq l-1$, then the following are true :*

   (a) *The experiment $(\bar{\mu}_{l-i}^t, \bar{\mu}_{l-i-1}^t)$ is optimal i.e*

   $$(\bar{\mu}_{l-i}^t, \bar{\mu}_{l-i-1}^t) \gtrsim_{(\mu,\pi,t)} \tau \text{ for all } \tau \in \Gamma(\mu)$$

   (b) *If $\pi = \bar{\pi}_l$ and $i = 0$, then*

   $$(\bar{\mu}_l^t, \bar{\mu}_{l-1}^t) \sim_{(\mu,\pi,t)} (0, \bar{\mu}_{l-1}^t)$$

The precise closed-form of the function $V_S^t(., \pi)$ is provided in the appendix section 5.3

Proof. Can be found in the appendix. The main arguments follow from Claim 3 in section 5.2 in the appendix. □

Note that the above proposition establishes the optimality of the seller's strategy (this is established in detail in the appendix section 5.3) at every triple (μ, π, t) such that $\pi \in [\bar{\pi}_l, \bar{\pi}_{l+1})$ where $0 \leq l < t$. Hence, the only cases that remain are $\pi = 1$ and $\pi \in [\bar{\pi}_t, 1)$. The former case $\pi = 1$ is straightforward to establish given buyer v_1's strategy[11]. Hence, one need only check for the remaining case $\pi \in [\bar{\pi}_t, 1)$. The following proposition describes the properties of the value function for this case and exhibits optimality of the seller's strategy :

Proposition 4. *Let $t > 0$, $\pi \in [\bar{\pi}_t, 1)$ and $V_S^t(\mu, \pi)$ be the value function of the seller in $G^T(\mu, \pi)$ defined by the above strategy profile.*

1. *The function $V_S^t(., \pi) : [0, 1] \to \mathbb{R}_+$ is non-decreasing, right-continuous and piecewise affine with the following set of break points :*

   $$\{\bar{\mu}_t^t, \bar{\mu}_{t-1}^t, ..., \bar{\mu}_1^t, \bar{\mu}_0^t\}$$

2. *Suppose the game is at a stage where (μ, π, t) such that $\mu < \bar{\mu}_l^t$. Let $\gtrsim_{(\mu,\pi,t)}$ be the seller's preference over experiments in $\Gamma(\mu)$. The following are true :*

   (a) *The experiment $(0, \bar{\mu}_t^t)$ is optimal i.e*

   $$(0, \bar{\mu}_t^t) \gtrsim_{(\mu,\pi,t)} \tau \text{ for all } \tau \in \Gamma(\mu)$$

   (b) *Moreover, the following relationship holds :*

[11] This case is essentially the same as there being no uncertainty about the buyer's type.

$$(0, \bar{\mu}_t^t) \succsim_{(\mu,\pi,t)} (0, \bar{\mu}_{t-1}^t) \succsim_{(\mu,\pi,t)} \dots \succsim_{(\mu,\pi,t)} (0, \bar{\mu}_1^t) \succsim_{(\mu,\pi,t)} (0, \bar{\mu}_0^t)$$

(c) *If $\pi = \bar{\pi}_l$, then we have :*

$$(0, \bar{\mu}_t^t) \sim_{(\mu,\pi,t)} (0, \bar{\mu}_{t-1}^t).$$

3. *Suppose the game is at a stage where (μ, π, t) such $\mu \in [\bar{\mu}_{t-i}^t, \bar{\mu}_{t-i-1}^t)$ with $0 \leq i \leq t-1$, then the following are true :*

   (a) *The experiment $(\bar{\mu}_{t-i}^t, \bar{\mu}_{t-i-1}^t)$ is optimal i.e*

   $$(\bar{\mu}_{t-i}^t, \bar{\mu}_{t-i-1}^t) \succsim_{(\mu,\pi,t)} \tau \text{ for all } \tau \in \Gamma(\mu)$$

   (b) *If $\pi = \bar{\pi}_t$ and $i = 0$, then*

   $$(\bar{\mu}_t^t, \bar{\mu}_{t-1}^t) \sim_{(\mu,\pi,t)} (0, \bar{\mu}_{t-1}^t)$$

The precise closed-form of the function $V_S^t(., \pi)$ is provided in the appendix section 5.3

Proof. Can be found in the appendix. The main arguments are completely analogous to the proof of Proposition 4. A discussion can be found in the appendix. □

4.0.3 Main Results

Propositions 3 and 4 essentially demonstrate the optimality of the seller's strategy. Now we turn our attention to the optimality of the buyer's strategy. This shall be relatively less involved and we contain its proof within the proof of our first main result below. The result states that the strategy specification provided above constitutes an equilibrium and highlights important qualititative features of the equilibrium for the T period game :

Proposition 5. *The following hold true :*

1. (*Equilibrium*) The strategy specification along with specified off-path beliefs constitutes an equilibrium.

2. (*Experiments*) On path, the seller uses a two-message experiments $\{m_0, m_1\}$ which only garbles the bad match. The message m_0 favors the good match and the m_1 favours the bad match. Moreover, if $T \geq 3$ and $\pi_0 \in [\bar{\pi}_2, \bar{\pi}_T)$ then second period onwards, the experiments chosen by the seller become increasingly informative in the Blackwell sense.

3. (*Rounds of Communication*) : If $\pi_0 \in [\bar{\pi}_l, \bar{\pi}_{l+1})$, then communication takes place for atmost $l+1$ periods. If $\pi_0 \in [\bar{\pi}_t, 1)$, then communication takes place for atmost t periods.

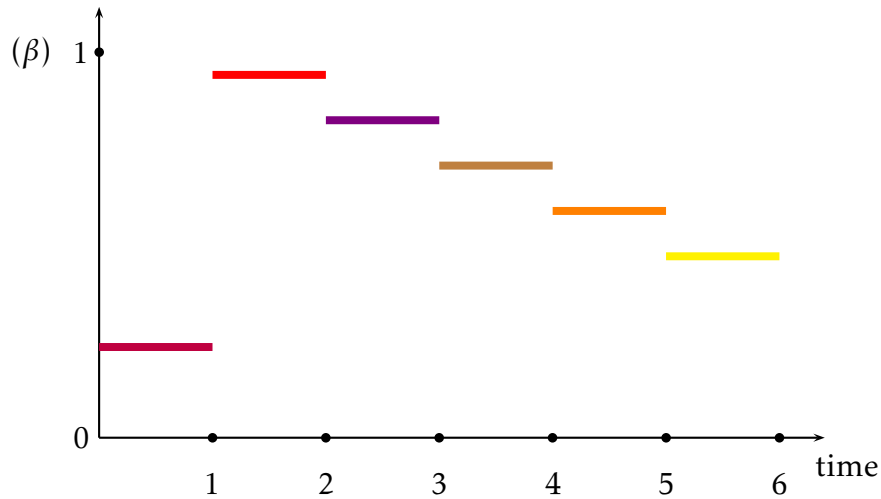
4. (*Learning the type*) If $\pi \geq \bar{\pi}_1$, then conditional on receiving m_0 perpetually, the seller learns the buyer's type.

Proof. Can be found in the appendix. □

Let us discuss the results derived above. Part 1) establishes that the strategies constructed constitute an equilibrium. The buyer's incentive constraints involve comparing the expected payoff obtained from buying in the current period with the value of future information. If $\pi_0 \geq \bar{\pi}_1$, then buyer v_1 is always kept indifferent between buying and not buying on the equilibrium path. A consequence of this indifference is that the value of future information at any stage simply equals the value of the experiment obtained in the next round (since buyer v_1 would be indifferent again tomorrow) upon not buying.

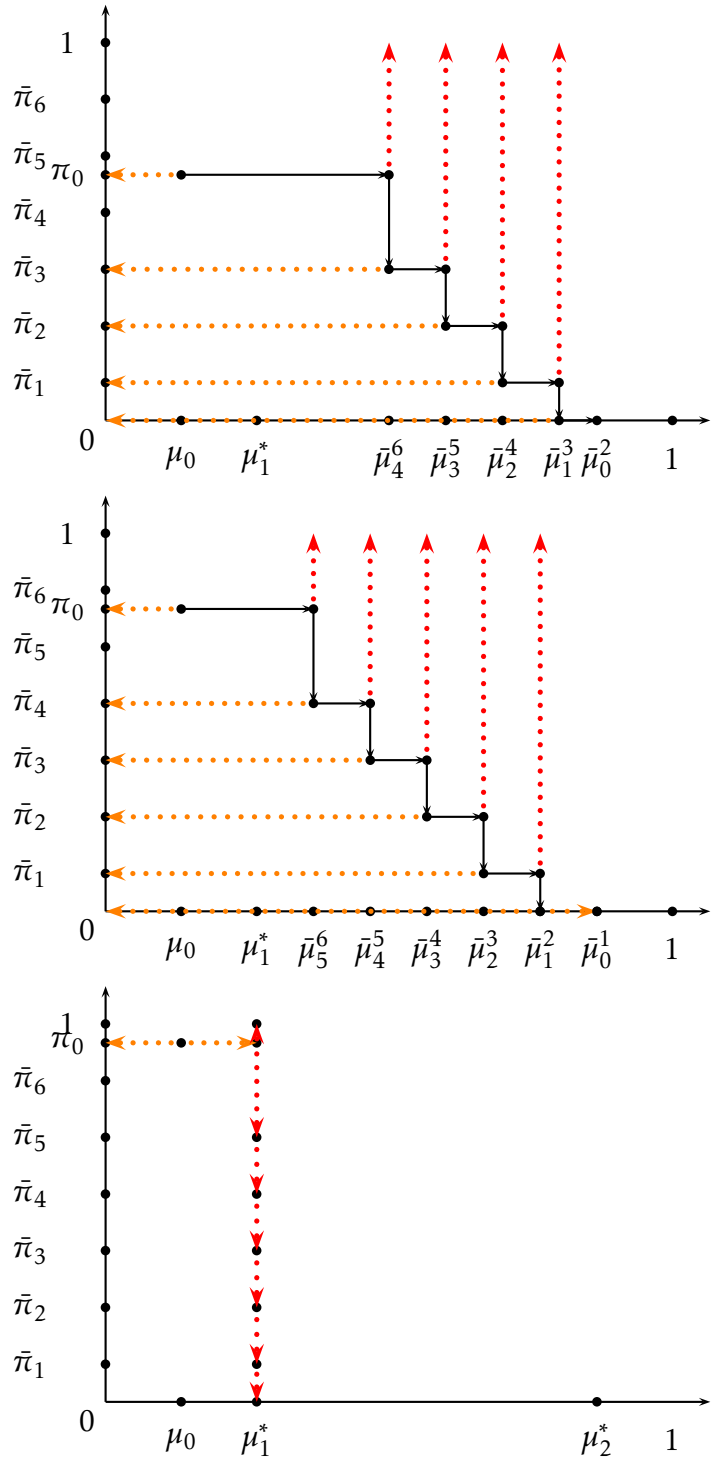
Consider now part 2). The fact that the seller chooses experiments which use only two messages essentially follows from the fact that Ω contains only two elements namely $\omega = G$ and $\omega = B$. In order to communicate the state ω, it suffices to use a message m_0 (good signal) to support $\omega = G$ and m_0 (bad signal) to support $\omega = B$[12]. The fact that the seller only wants to garble the bad match is intuitive. If the match is indeed good, the seller would want the buyer to know this is the case. Hence, he would wish to choose experiments which yield m_0 with probability one under $\omega = G$. On the other hand, if the match is bad, the seller would want the buyer to be kept in the dark. As a result, he would choose experiments which report m_0 with some positive probability under $\omega = B$. The second statement made in the above proposition could be argued thusly : Since the buyer is always indifferent, the expected payoff from buying at the current belief equals the value of the experiment obtained subsequently by not buying. Now if this experiment yields a good signal the belief goes up again to a higher belief. The buyer is again indifferent between buying and not buying. However, now the expected payoff from buying at the higher belief is greater than the expected payoff from the buying at the previous belief. From indifference, this implies that the value of the experiment obtained by not buying at the higher belief is greater than the value of the experiment obtained at the previous belief. Now implies that the latter experiment cannot be more informative than the former in the blackwell sense. But given that experiments have the structure that they reveal the bad state perfectly, we can further argue that the latter experiment must necessarily be less informative in blackwell sense. This is also equivalent to saying that the probability of reporting m_0 second period onwards strictly decreases over time. The following figure plots this probability (β) against time for $T = 6$ and $\pi_0 \in [\bar{\pi}_5, \bar{\pi}_6)$:

[12]See for example Proposition 1 in Kamenica-Gentzkow (2011)



Now consider part 3). This states that if $\pi_0 \in [\bar{\pi}_l, \bar{\pi}_{l+1})$, the buyer and the seller will communication for atmost $l+1$ periods. Note that this does not depend on the horizon length T. If $T \leq l+1$, then the statement if trivial. However, it continues to hold even for any $T > l+1$. Hence, horizons could be arbitrarily long and still not influence how long players engage in exchanging information. The reason behind this phenomenon is as follows : If $\pi_0 \in [\bar{\pi}_0, \bar{\pi}_1)$, we know that players will communication for only one period. The seller provides an experiment at the beginning (namely $(0, \mu_2^*)$). If μ_2^* is realised, both buyer types would buy and if 0 is realised, both buyer types would not buy. Now if $\pi_0 \in [\bar{\pi}_1, \bar{\pi}_2)$, experiment $(0, \bar{\mu}_1^T)$ would be chosen. If $\bar{\mu}_1^T$ is realised and a decision to not buy is observed, the seller's belief about the buyer would fall to $\pi = 0 = \bar{\pi}_0 \in [\bar{\pi}_0, \bar{\pi}_1)$ at which we know only one period of communication takes place. Hence, starting from π_0 only two periods of communication ensue. Proceeding in this manner, we get that if $\pi_0 \in [\bar{\pi}_l, \bar{\pi}_{l+1})$, if experiments provided by the seller keep shifting posteriors about ω upwards, perpetual decisions to not buy would make π fall from π_0 to $\bar{\pi}_{l-1}$, then from $\bar{\pi}_{l-1}$ to $\bar{\pi}_{l-2}$ as so on till we reach $\pi = 0 = \bar{\pi}_0$.

It is precisely due to the above dynamics that part 4) follows as well. Since buyer v_1 is indifferent at every step (provided experiments are successful and μ goes upwards) any decision to not buy before the $l+1$-th period of communication would reveal that it is indeed v_1. At the penultimate stage of communication i.e the l-th period, the types are revealed perfectly. If the buyer has not bought, then he is revealed to be v_2 and the seller provides a final experiment attempting to push beliefs to μ_2^* and persuade him to buy. The following diagrams illustrate the dynamics of the belief pair (μ, π) for the case $T = 6$ and demonstrates the dynamics mentioned above. The cases considered are $\pi_0 \in [\bar{\pi}_4, \bar{\pi}_5)$, $\pi_0 \in [\bar{\pi}_5, \bar{\pi}_6)$ and $\pi_0 \in [\bar{\pi}_6, \bar{\pi}_1)$:



We now consider the case when $T \to \infty$. In this case, certain new features emerge. Our results demonstrate that with very long horizons, bulk of the information is provided in the first period (in a manner we shall just make precise). Additionally, a *reputation effect* arises stating that if π_0 is close to 1 and μ_2^* is close to 1, the experiment chosen in the first period is close to one which yields full disclosure. We present the result as follows :

Proposition 6. *The following statements hold true for the given equilibrium :*

1. *For every $\epsilon > 0$ there exists $\bar{T} \in \mathbb{N}$ such that for any $T \geq \bar{T}$, the experiment chosen in $\hat{G}^T(\mu_0, \pi_0)$ is of the form $(0, \mu')$ where $0 < \mu_0 < \mu' < \mu_2^*$ and*

$$|\mu' - \mu_2^*| < \epsilon$$

   Moreoever, all future posteriors are either equal to zero or equal to $\mu'' \in [\mu', \mu_2^*]$

2. *For any $\epsilon > 0$ and $1 - \epsilon < \pi_0, \mu_2^* < 1$, there exists $\bar{T} \in \mathbb{N}$ such that for any $T \geq \bar{T}$, the experiment chosen in $\hat{G}^T(\mu_0, \pi_0)$ is of the form $(0, \mu')$ such that*

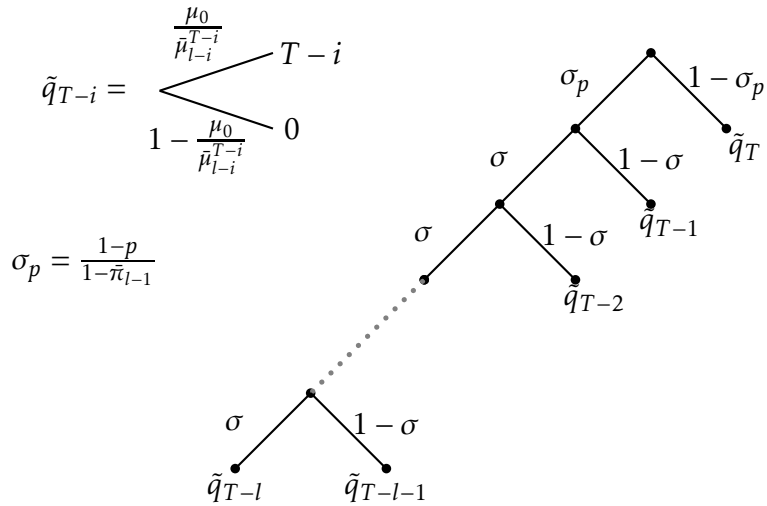
$$1 - \epsilon < \mu' < 1$$

   i.e the experiment $(0, \mu')$ is close to full disclosure.

Proof. Can be found in the appendix. □

With to regard part 1 of the above proposition, an alternative way of measuring information disclosed in the first period as a proportion of total information disclosed ever time, one can consider the notion of reduction in entropy. This would be done as follows : calculate the reduction in entropy from the experiment chosen in the first period and divide by the total reduction in entropy from experiments supplies over time (conditional on a_1 being played perpetually). One can show that this ratio converges to 1 as $T \to \infty$.

Part 2 pertains to the reputation effect highlighted above and earlier in the introduction. To fully appreciate the result consider the case when $\mu_0 \approx \mu_1^*$, $\pi_0 \approx 1$ and $\mu_2^* \approx 1$. Suppose $\mu_0 \approx \mu_1^*$, if it were the case that $\pi_0 = 1$, then the seller would choose experiment $(0, \mu_1^*)$ which would be virtually non-informative. However, with $\pi_0 \approx 1$ what obtains is the polar opposite. As $T \to \infty$, the experiment chosen in the first period is $(0, \mu')$ where $\mu' \approx \mu_2^* \approx 1$ i.e. the experiment $(0, \mu')$ is close to full disclosure. This presents a discontinuity in the model with a drastic impact on how information is provisioned by the seller. Slight uncertainty about the buyer's type can have a big influence. This phenomenon is reminiscent of features observed in some canonical dynamic models of reputation as in Kreps-Wilson(1982)[15], Milgrom-Roberts(1982)[20], Kreps-Milgrom-Roberts-Wilson(1982) [16]and Fudenberg-Levine(1989)[12].



Delay in Persuasion and Stopping Time : Lastly, note that the equilibrium we constructed is such that on path, the buyer randomises between buying and not buying. This happens as follows. Note first that in any stage if the good is bought, it must be the case that $\mu \geq \mu_1^*$. Now, if $\mu_1^* \leq \mu < \mu_2^*$ and b is observed by the seller, then the buyer is revealed to be v_1 If $\mu = \mu_2^*$ then both buyer types play b. Hence, there is concession/stopping time associated with persuasion. One can explicitly compute this stopping time and can represent it as follows : The stopping time has the following representation. Every period, "something" can arrive with some probability (this is $1-\sigma_p$ for the first period and $1-\sigma$ for all subsequent periods). If arrival takes places when $T-i$ periods are left, we obtain a lottery which with some probability leads to immediate persuasion and with complementary probability leads to persuasion never occuring. This phenomenon of persuasion in the present model bears resemblence with the notion of agreement or concession in models of bargaining with incomplete information as in Fudenberg-Levine-Tirole (1985)[13], Chatterjee-Samuelson(1987)[7] and Chatterjee-Samuelson(1988)[8].

5 Discussion

5.1 Generality of the Model

The model presented above involved an interaction between a seller and a buyer. The objective of the seller was to persuade the buyer to buy a durable product at a given price. This problem belongs to a broader class of dynamic Bayesian persuasion problems and qualifies as a special case. However, the analysis provided above carries over exactly to even this broader class without any loss of generality. Let us describe this class of problems and reason why the analysis above applies.

There are two parties namely a Sender and a Receiver. The Sender wishes to persuade the Receiver to take a desired *irreversible* action a_d over a status quo action a_s. Taking an action $a \in \{a_d, a_s\}$ yields a per-period utility that depends on the state of nature $\omega \in \{G, B\}$ and the Receiver's *type* via utility function $u(a, \omega; v)$ where $v \in \mathcal{V} := \{v_1, v_2\}$. These utilities are such that $u(a_d, G; v) > u(a_d, G; v)$ and $u(a_s, B; v) > u(a_s, B; v)$ for all $v \in \mathcal{V}$. Hence, if $\omega = G$ the Receiver would strictly prefer taking the desirable action a_d and if $\omega = B$, the Receiver would strictly prefer a_s. On the other hand, the Sender only wants the desired action to be taken i.e he obtains a per-period utility of $v(a) = K\mathbb{I}_{\{a=a_d\}}$ whenever action a is taken where $K > 0$. We assume that both players wish to maximise *total payoffs* i.e life-time utilities are given by $\sum_{t=1}^{T} u(a_t, \omega; v)$ and $\sum_{t=1}^{T} v(a_t)$ for a given sequence of actions $\{a_t\}_{t=1}^{T}$.

Notice first that the model presented above is special of the model just described. In the buyer-seller framework, $\mathcal{V} \subseteq \mathbb{R}_{++}$; $u(a_d, G; v) = v - q > 0 > -q = u(a_d, B; v)$; $u(a_s, G; v) = u(a_s, B; v) = 0$ and $K = q$. We now argue that analysis presented above carries over exactly to the more general framework. Note that all quantities used to define strategies namely $\bar{\mu}_l^t$, $\bar{\pi}_l$ and σ were defined completely in terms of the buyer's threshold μ_1^* and μ_2^*. Now we can analogously define these thresholds in the general case. For Receiver type v_i, μ_i^* will be defined as the belief μ at which he would be indifferent between a_d and a_s if no further information is provided. Hence μ_i^* is the unique solution to the equation $\mu u(a_d, G; v_i) + (1-\mu)u(a_d, B; v_i) = \mu u(a_s, G; v_i) + (1-\mu)u(a_s, B; v_i)$. Suppose without loss of generality that $\mu_1^* < \mu_2^*$. Now once we define these thresholds, the Sender's strategy would correspond to the seller's strategy as described above and the Receiver's strategy would correspond to the buyer's strategy as described above. Notice that given the Receiver's strategy, the Sender's strategy would be optimal since it only involves deriving his value function $V^t(\mu, \pi)$ and using its concavification. Note however that this value function is simply the value function we derive for the seller (see appendix) multiplied by a positive scalar $\frac{K}{q}$. This guarantees optimality of the Sender's strategy. Now to check that the Receiver's strategy would be optimal it shall be helpful to establish the following claim on state-dependent utility functions :

Claim 2. *: If u, u' have the same threshold μ^*, then there exists $\kappa > 0$ and $c : \Omega \to \mathbb{R}$ such that $u'(a, \omega) = \kappa u(a, \omega) + c(\omega)$*

The above statement is established in the appendix. Let us now argue that the Receiver's incentive constraints would be equivalent to that of the buyer. Consider the ex-

pression $\kappa u(a,\omega) + c(\omega)$ and consider the incentives of a Receiver with this utility function. At any given stage when the belief about ω is μ, consider the Receiver's comparison of long run payoffs from a_d and a_s. Since $c(\omega)$ does not depend on actions, and since the belief process (about ω) is a martingale a payoff equal to $\mathbb{E}_\mu[c(\omega)]$[13] is added every period. Hence, only the part $\kappa u(a,\omega)$ relevant for comparison of actions. However note that the comparison would be same for a Receiver with $u(a,\omega)$. This allows us to focus on the given payoff specification for the buyer without any loss of generality.

5.2 Non-Durable Goods, Repeated Actions and Dynamic Public Persuasion

Suppose in the general framework described above, the action a_d was not assumed to be irreversible. Hence, after a choice of a_d, the Receiver can choose to play any action $a \in \{a_d, a_s\}$. This corresponds to a situation where the actions of the Receiver are chosen *repeatedly* and influences the welfare of both the Sender and the Receiver via payoffs described in the above section. In the buyer-seller framework, this would correspond to the case of a non-durable good which could be consumed every period. It turns out that even in this class of settings, the equilibrium strategies described as above can still be supported. Of course, the additional consideration that needs to made is describing behaviour after histories which include the choice of a_d. This turns out to be straightforward to achieve.

The Markovian strategies prescribing actions on triples (μ, π, t) for the Receiver and the Sender would remain the same. The off-path beliefs would be need to be specified more carefully. At (μ, π, t), if the Sender observes that a_d was chosen when it should not have been (by either Receiver types), the Sender believes that it was the Receiver v_1 (justified by him having a lower threshold and being more likely to make the deviation). This changes the state of the game to $(\mu, 1, t-1)$ at which the Sender plays according to his Markovian strategy.

This class of problems can be thought of as dynamic versions of the public persuasion mechanisms considered in Rayo-Segal (2010) and Kolotolin et. al. (2015). Here, the Sender lacks long term commitment power. In constrast to the static case, we see that even low threshold types are able to obtain a large amount of information. If thresholds were known, the solution to the static and dynamic problems would be identical.

[13] This is because the marginal of a belief martingale at any time point has mean equal to the prior μ

5.3 Infinite Horizon with Discounting

In this paper we consider a finite horizon setting. Our analysis and results would continue to hold even in a setting in infinite horizon with discounting. The payoffs in the model would be as follows : If at any time point t a buying decision is made, the buyer would get $\delta^t(v_i \mathbb{I}_{\{\omega=G\}} - p)$ and the seller would obtain $\delta^t p$. However, in this case we would be interested in analysing stationary equilibria of the game i.e PBE in strategies which only depend on beliefs about the match quality and buyer valuation. The construction of strategies would now involve cut-offs $\{\bar{\pi}_l^\infty\}_{l\geq 1}$ and $\{\bar{\mu}_l^\infty\}_l$. Here we define $\bar{\mu}_l^\infty := \dfrac{1}{(1-\delta^l)\left(\dfrac{1}{\mu_1^*}\right) + \delta^l\left(\dfrac{1}{\mu_2^*}\right)}$ and have that $\lim_{l\to\infty} \bar{\pi}_l = 1$ and $\bar{\pi}_{l+1} > \bar{\pi}_l$[14].

Equilibrium behaviour would be as follows : If $\pi_0 \in [\bar{\pi}_l, \bar{\pi}_{l+1})$, then in the first round experiment $(0, \bar{\mu}_l^\infty)$ would be provided. If $\bar{\mu}_l^\infty$ is realised and the good is not bought, then experiment $(0, \bar{\mu}_{l-1}^\infty)$ is provided. Hence, if beliefs about $\omega = G$ keep going upwards and the good is not bought perpetually, then experiments $\{(0, \bar{\mu}_{l-i}^\infty)\}_{0\leq i\leq l}$ are chosen. The high valuation buyer would randomise every period between buying and not buying and the low valuation buyer would choose to buy if and only if $\mu \geq \mu_2^*$.

6 Conclusion

We analysed a dynamic Bayesian persuasion problem involving a seller and a buyer. The main focus of our analysis is studying how the seller could optimally provide information over time to maximise the chances of a purchase. A crucial feature of our model is the fact that the buyer is privately informed of his valuation for a relevant product. The presence of this feature distinguishes our work from the prior literature on dynamic Bayesian persuasion. We observed that the effect of the private information is very strong. It allows high valuation buyers to extract information far more than they would have got if their valuation was known. Moreover, the bulk of this information is provided at the beginning of the interaction. We also highlight important qualitative features of equilibrium behaviour. For future work, many aspects can be investigated further. We highlight here two such aspects : 1) An analysis of a larger set of types (valuations) would be useful in understanding further the role of private information in the framework presented here 2) Another line of study would be analysing our setting in the presence of transfers and how it would affect equilibrium features of the model.

[14] In constrast to the finite horizon case, it is difficult to provide a closed form expression for $\bar{\pi}_k$.

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7 Appendix

7.1 Concavification

We now formally define the concavification operator[15] introduced in section 4.0.1. :

[15] See also Aumann-Maschler (1995) Ch. 1 pp. 23

Definition 4. *Let $f : [0,1] \to \mathbb{R}$ be a bounded function and consider the set of concave functions $\mathcal{F}^+ := \{g \in \mathbb{R}^{[0,1]} : g$ is concave and $g(x) \geq f(x) \forall x \in [0,1]\}$ which pointwise dominate f. The concavification of f, denoted as $cav[f(x)]$ is defined as :*

$$cav[f(x)] := \inf_{g \in \mathcal{F}^+} g(x) \tag{12}$$

Note that $\text{cav}[f(x)]$ is itself concave since it is the pointwise infimum of concave functions. In order to establish Proposition 8, the following result on non-decreasing, right continuous and piecewise affine functions is useful :

Claim 3. *Let $f : [0,1] \to \mathbb{R}_+$ be a function and $\{\bar{x}_0, \bar{x}_1, \bar{x}_2, ..., \bar{x}_n, \bar{x}_{n+1}\} \subseteq [0,1]$ be a set of points such that*

$$0 = \bar{x}_0 < \bar{x}_1 < \bar{x}_2 < ... < \bar{x}_n < \bar{x}_{n+1} = 1$$

where $n \geq 2$ and additionally suppose that the following properties hold true :

1. *(Zero at zero) $f(0) = 0$*
2. *(Constant at the end) For all $x \in [\bar{x}_n, 1]$ we have $f(x) = K$; where $K > 0$ is constant*
3. *(Piecewise Affine) f is positively affine on each segment $[\bar{x}_i, \bar{x}_{i+1})$, $0 \leq i \leq n-1$, i.e there exist $a_i \geq 0$, $b_i \in \mathbb{R}$ such that*

$$f(x) = a_i x + b_i \text{for all } x \in [\bar{x}_i, \bar{x}_{i+1}) \tag{13}$$

4. *(Jumps) For $\bar{x}_1$ and for all $i \in \{2, .., n\}$ it is true that :*

$$f(\bar{x}_1) \geq \lim_{y \uparrow \bar{x}_1} f(y) \tag{14}$$

$$f(\bar{x}_i) > \lim_{y \uparrow \bar{x}_i} f(y) \tag{15}$$

5. *(Decreasing averages for $i \geq 2$) For $\bar{x}_1$ and for all $i \in \{2, .., n\}$ it is true that :*

$$\frac{f(\bar{x}_1)}{\bar{x}_1} < \frac{f(\bar{x}_2)}{\bar{x}_2} \tag{16}$$

$$\frac{f(\bar{x}_i)}{\bar{x}_i} \geq \frac{f(\bar{x}_{i+1})}{\bar{x}_{i+1}} \tag{17}$$

6. *(Decreasing slopes for $i \geq 2$) For all $i \in \{2,..,n-1\}$ it is true that :*

$$\frac{f(\bar{x}_{i+1}) - f(\bar{x}_i)}{\bar{x}_{i+1} - \bar{x}_i} \geq \frac{f(\bar{x}_{i+2}) - f(\bar{x}_{i+1})}{\bar{x}_{i+2} - \bar{x}_{i+1}} \tag{18}$$

Then, the following are true :

(a) *f is non-decreasing*

(b) *For all $x < \bar{x}_2$, we have :*

$$cav[f(x)] = \frac{x}{\bar{x}_2} f(\bar{x}_2) \tag{19}$$

(c) *For all $i \in \{2,..,n\}$ and $x \in [\bar{x}_i, \bar{x}_{i+1}]$ we have :*

$$cav[f(x)] = \left[\frac{x - \bar{x}_i}{\bar{x}_{i+1} - \bar{x}_i}\right] f(\bar{x}_{i+1}) + \left[\frac{\bar{x}_{i+1} - x}{\bar{x}_{i+1} - \bar{x}_i}\right] f(\bar{x}_i) \tag{20}$$

(d) *For all $i \in \{2,..,n\}$:*

$$cav[f(\bar{x}_i)] = f(\bar{x}_i) \tag{21}$$

(e) *For all $x \in [\bar{x}_{n+1}, 1]$:*

$$cav[f(x)] = K \tag{22}$$

Further, if any function f satisfies properties 1-6, and we define a function $f' = cf$ for some $c > 0$, then f' satisfies properties 1-6 as well.

Proof. Part a) follows trivially from 3 and 4. We begin by proving part b). Define the function g on $[0,1]$:

$$g(x) := \frac{x}{\bar{x}_2} f(\bar{x}_2)$$

Step 1 : We will first show that $g(x) \geq f(x)$ for $x \in [0,1]$. Clearly $g(0) = 0 = f(0)$. Suppose $x \in (0, \bar{x}_1)$. Now since $f(0) = 0$ and since f is affine on $[0, \bar{x}_1)$, it follows that $\frac{f(x)}{x} = \frac{f(y)}{y}$ for all $y \in (0, \bar{x}_1)$.Hence, it follows that $\frac{f(x)}{x} = \lim_{y \uparrow \bar{x}_1} \frac{f(y)}{y} \leq \frac{f(\bar{x}_1)}{\bar{x}_1} < \frac{f(\bar{x}_2)}{\bar{x}_2}$ (using 4 and 5) which in turn implies that $f(x) < g(x)$. Now consider $x \in [\bar{x}_1, \bar{x}_2)$. It is clear that $f(\bar{x}_1) < g(\bar{x}_1)$ from 5 so let us consider $x \in (\bar{x}_1, \bar{x}_2)$. Suppose for contradiction we have that $\frac{f(x)}{x} > \frac{f(\bar{x}_2)}{\bar{x}_2}$.

Using 5 and the fact that f is affine on $[\bar{x}_1, \bar{x}_2)$, we invoke the Intermediate Value Theorem to find $x^* \in (\bar{x}_1, x)$ such that $\frac{f(x^*)}{x^*} = \frac{f(\bar{x}_2)}{\bar{x}_2}$. This implies that

$$\frac{f(x) - f(x^*)}{x - x^*} > \frac{f(x^*)}{x^*} = \frac{f(\bar{x}_2) - f(x^*)}{\bar{x}_2 - x^*}$$

Now since f is affine on $[\bar{x}_1, \bar{x}_2)$, it must be the case that for all $y \in (x^*, \bar{x}_2)$, we have $\frac{f(y) - f(x^*)}{y - x^*} = \frac{f(x) - f(x^*)}{x - x^*}$. Now, using 4 we get that $\lim_{y \uparrow \bar{x}_2} \frac{f(y) - f(x^*)}{y - x^*} = \frac{f(x) - f(x^*)}{x - x^*} < \frac{f(\bar{x}_2) - f(x^*)}{\bar{x}_2 - x^*}$. But this contradicts what we established above. Consider now $x \in [\bar{x}_i, \bar{x}_{i+1})$ where $i \geq 2$. From 5 we know $\frac{f(\bar{x}_i)}{\bar{x}_i} \geq \frac{f(\bar{x}_{i+1})}{\bar{x}_{i+1}}$ which implies $\frac{f(\bar{x}_{i+1}) - f(\bar{x}_i)}{\bar{x}_{i+1} - \bar{x}_i} \leq \frac{f(\bar{x}_i)}{\bar{x}_i}$. Using 3 and 4, we hence get $\frac{f(x) - f(\bar{x}_i)}{x - \bar{x}_i} < \frac{f(\bar{x}_{i+1}) - f(\bar{x}_i)}{\bar{x}_{i+1} - \bar{x}_i} \leq \frac{f(\bar{x}_i)}{\bar{x}_i}$. Now from 5, we get $\frac{f(x)}{x} < \frac{f(\bar{x}_i)}{\bar{x}_i} \leq \frac{f(\bar{x}_2)}{\bar{x}_2}$ which implies $f(x) < g(x)$. Finally consider $x \in [\bar{x}_n, 1]$. Using 2 and 5, we get :

$$\frac{f(x)}{x} = \frac{K}{x} \leq \frac{K}{\bar{x}_n} = \frac{f(\bar{x}_n)}{\bar{x}_n} \leq \frac{f(\bar{x}_2)}{\bar{x}_2}$$

Hence, $f(x) \leq g(x)$.

Step 2 : We show (b) holds true. Since $g \geq f$ and since g is linear, it follows that $g(x) \geq \text{cav}[f(x)]$ for all $x \in [0,1]$. Hence, it suffices to show that $g(x) \leq \text{cav}[f(x)]$ for all $x \in [0, \bar{x}_2]$. First observe that $\text{cav}[f(\bar{x}_2)] \leq \frac{\bar{x}_2 f(\bar{x}_2)}{\bar{x}_2} = f(\bar{x}_2)$. But by definition we have $\text{cav}[f(\bar{x}_2)] \geq f(\bar{x}_2)$. Hence $g(\bar{x}_2) = \text{cav}[f(\bar{x}_2)]$. Also, by a similar argument, $g(0) = 0 = \text{cav}[f(0)]$. Now, let $x \in (0, \bar{x}_2)$. Setting $\lambda = 1 - \frac{x}{\bar{x}_2}$ we now have :

$$\begin{aligned}
g(x) &= g(\lambda(0) + (1-\lambda)(\bar{x}_2)) \\
&= \lambda g(0) + (1-\lambda) g(\bar{x}_2) \\
&= \lambda \text{cav}[f(0)] + (1-\lambda)\text{cav}[f(\bar{x}_2)] \\
&\leq \text{cav}[f(\lambda(0) + (1-\lambda)\bar{x}_2)] \\
&\leq \text{cav}[f(x)]
\end{aligned}$$

This completes the proof of part b).

Now consider part c). First note that for any two affine functions $f', g' : [a,b] \to \mathbb{R}$ such

that $f'(a) \geq g'(a)$, $f'(b) \geq g'(b)$, it is true that $f'(x) \geq g'(x)$ for all $x \in [a,b]$. Now define the following family $\{h^i\}_{1 \leq i \leq n}$ of affine functions

$$h^1(x) = \frac{xf(\bar{x}_2)}{\bar{x}_2} \tag{23}$$

$$h^i(x) = \left[\frac{x - \bar{x}_i}{\bar{x}_{i+1} - \bar{x}_i}\right] f(\bar{x}_{i+1}) + \left[\frac{\bar{x}_{i+1} - x}{\bar{x}_{i+1} - \bar{x}_i}\right] f(\bar{x}_i) \tag{24}$$

where $2 \leq i \leq n$. Note that h^i is simply the function g that was defined in the proof of part b). We now proceed using the following steps.

Step 1 : We show for each $2 \leq i \leq n$, it is the case that $h^i(x) \geq f(x)$ for all $x \in [\bar{x}_i, \bar{x}_{i+1}]$. Consider first the case $2 \leq i \leq n-1$. Consider the affine functions $f^i(x) := a_i x + b_i$ defined on $[\bar{x}_i, \bar{x}_{i+1}]$. Hence $f^i = f$ on $[\bar{x}_i, \bar{x}_{i+1})$. Now observe that $h^i(\bar{x}_i) = f^i(\bar{x}_i)$ and using 4 we have $h^i(\bar{x}_{i+1}) = f(\bar{x}_{i+1}) > f^i(\bar{x}_{i+1})$. Since, h^i and f^i are both affine it follows that $h^i(x) \geq f(x)$ on $x \in [\bar{x}_i, \bar{x}_{i+1}]$. Now for the case $i = n$, since $f = K = h^i$ on $[\bar{x}_n, \bar{x}_{n+1}]$, the conclusion follows trivially.

Step 2 : For each $2 \leq i, j \leq n$ such that $j \neq i$, it is true that $h^i(x) \geq h^j(x)$ for all $x \in [\bar{x}_j, \bar{x}_{j+1}]$. Note that in order to show this, it suffices to show that for all $2 \leq i, k \leq n+1$ the following holds :

$$h^i(\bar{x}_k) \geq h^k(\bar{x}_k) \tag{25}$$

This is because the above relation would imply $h^i(\bar{x}_j) \geq h^j(\bar{x}_j)$ and $h^i(\bar{x}_{j+1}) \geq h^j(\bar{x}_{j+1})$ for all $1 \leq j \leq n$. Since, h^i, h^j are both affine, the desired result would follow. Hence, consider $2 \leq k \leq n$ and suppose first that $i < k$. By definition, we have $h^k(\bar{x}_k) = f(\bar{x}_k)$ and we have :

$$\begin{aligned}
h^i(\bar{x}_k) &= \left[\frac{\bar{x}_k - \bar{x}_i}{\bar{x}_{i+1} - \bar{x}_i}\right] f(\bar{x}_{i+1}) + \left[\frac{\bar{x}_{i+1} - \bar{x}_k}{\bar{x}_{i+1} - \bar{x}_i}\right] f(\bar{x}_i) \\
&= \left[\frac{f(\bar{x}_{i+1}) - f(\bar{x}_i)}{\bar{x}_{i+1} - \bar{x}_i}\right] \bar{x}_k + \left[\frac{\bar{x}_{i+1} f(\bar{x}_i) - \bar{x}_i f(\bar{x}_{i+1})}{\bar{x}_{i+1} - \bar{x}_i}\right] \\
&= \left[\frac{f(\bar{x}_{i+1}) - f(\bar{x}_i)}{\bar{x}_{i+1} - \bar{x}_i}\right] \bar{x}_k + \left[\frac{\bar{x}_{i+1} f(\bar{x}_i) - \bar{x}_i f(\bar{x}_i) + \bar{x}_i f(\bar{x}_i) - \bar{x}_i f(\bar{x}_{i+1})}{\bar{x}_{i+1} - \bar{x}_i}\right] \\
&= \left[\frac{f(\bar{x}_{i+1}) - f(\bar{x}_i)}{\bar{x}_{i+1} - \bar{x}_i}\right] (\bar{x}_k - \bar{x}_i) + f(\bar{x}_i) \\
&\geq f(\bar{x}_k) \\
&= h^k(\bar{x}_k)
\end{aligned}$$

where the last inequality follows from the fact that $\left[\frac{f(\bar{x}_{i+1})-f(\bar{x}_i)}{\bar{x}_{i+1}-\bar{x}_i}\right] \geq \left[\frac{f(\bar{x}_k)-f(\bar{x}_i)}{\bar{x}_k-\bar{x}_i}\right]$ which itself follows from 6. An analogous argument shows that this is also the case when $k < i$

Step 3 : We now show that for $1 \leq i \leq n$ it is true that $h^i(x) \geq f(x)$ for all $x \in [0,1]$. First consider $2 \leq i \leq n$. Note that Step 1 and Step 2 imply that this is the case for all $x \in [\bar{x}_2, 1]$. Hence, we wish to show it for the region $[0, \bar{x}_2]$. Recall from part b) that $h^1(x) \geq f(x)$ for all $x \in [0, \bar{x}_2]$. Therefore, it suffices to show that $h^i(0) \geq h^1(0)$ and $h^i(\bar{x}_2) \geq h^1(\bar{x}_2)$. The first inequality follows from the fact that $h^i(0) = \left[\frac{\bar{x}_{i+1}f(\bar{x}_i)-\bar{x}_i f(\bar{x}_{i+1})}{\bar{x}_{i+1}-\bar{x}_i}\right] \geq 0 = h^1(0)$ which itself follows from 5. The second inequality follows due to arguments presented in Step 2. Finally, note that part b) already establishes $h^1 \geq f$.

Step 4 : We now establish the statement in part c). Let $2 \leq i \leq n$. Since h^i is affine and $h^i \geq f$ it is the case that $h^i \geq \text{cav}[f]$. Moreover, note that $h^i(\bar{x}_i) = f(\bar{x}_i)$ and $h^i(\bar{x}_{i+1}) = f(\bar{x}_{i+1})$. Hence, $h^i(\bar{x}_i) = \text{cav}[f(\bar{x}_i)]$ and $h^i(\bar{x}_{i+1}) = \text{cav}[f(\bar{x}_{i+1})]$. Now consider $x \in [\bar{x}_i, \bar{x}_{i+1}]$ and let $\lambda \in [0,1]$ such that $x = \lambda\bar{x}_i + (1-\lambda)\bar{x}_{i+1}$. Then,

$$\begin{aligned}
h^i(x) &= h^i(\lambda(\bar{x}_i) + (1-\lambda)(\bar{x}_{i+1})) \\
&= \lambda h^i(\bar{x}_i) + (1-\lambda)h^i(\bar{x}_{i+1}) \\
&= \lambda\text{cav}[f(\bar{x}_i)] + (1-\lambda)\text{cav}[f(\bar{x}_{i+1})] \\
&\leq \text{cav}[f(\lambda(\bar{x}_i) + (1-\lambda)\bar{x}_{i+1})] \\
&\leq \text{cav}[f(x)]
\end{aligned}$$

Part d) follows trivially from parts b) and c). For part e) follows from part c) and 7 yielding $h^n(x) = K$ for all $x \in [\bar{x}_n, 1]$. Finally, that $f' = cf$ satisfies properties 1-6 follows straightforwardly. □

7.2 Proof of Proposition 3

Here we establish in detail the various properties satisfied by the seller's value function $V_S^t(\mu, \pi)$ as highlighted in the statement of Proposition 4. We shall do so by showing that $V_S^t(., \pi)$ satisfies conditions 1-6 of Claim 2 and that the satisfaction of these conditions via its conclusions a)-e) would in turn imply all the properties.

Note that if ever the good is bought by the buyer at stage t, the seller obtains a lifetime utility equal to pt. Hence, the value function $V_S^t(.,\pi)$ includes a factor of p within it. In deriving the properties, we shall for convenience work with continuation values defined by scaling the value function by a factor $\frac{1}{q}$ i.e $V^t(\mu,\pi) := \frac{V_S^t(\mu,\pi)}{q}$. This avoids repeatedly writing the factor q when deriving the value function. In order to obtain the original value function we would need only multiply by the factor p again. Note from Claim 3 that properties 1-6 would be preserved by positive scalar multiplication. The conditions stated in the Proposition 8 would be preserved as well since the breaks will remain the same as well as the relation $\gtrsim_{(\mu,\pi,t)}$. Similarly define, $\hat{V}^t(\mu,\pi) = \frac{\hat{V}_S^t(\mu,\pi)}{p}$. We now derive the function $V^t(\mu,\pi)$.

Value of experiment $(0,\bar{\mu}_k^t)$: We first begin by computing the value of experiments of the form $(0,\bar{\mu}_k^t)$ where $0 < k \leq t$ when $\pi \geq \bar{\pi}_{k-1}$ and $\mu \leq \bar{\mu}_k^t$. This value is equal to :

$$\frac{\mu}{\bar{\mu}_k^t} V^t(\mu_k^t,\pi) = \frac{\mu}{\bar{\mu}_k^t}\left[\left(\frac{\pi-\bar{\pi}_{k-1}}{1-\bar{\pi}_{k-1}}\right)[t] + \left(\frac{1-\pi}{1-\bar{\pi}_{k-1}}\right)\hat{V}^{t-1}(\bar{\mu}_k^t,\bar{\pi}_{k-1})\right] \tag{26}$$

Hence, we need only ascertain $\hat{V}^{t-1}(\bar{\mu}_k^t,\bar{\pi}_{k-1})$. In order compute this value, it shall be useful to keep track of the belief process $\{(\mu^s,\pi^s)\}_s$ starting from $(\bar{\mu}_k^t,\bar{\pi}_{k-1})$. Note from the strategy specification of the seller and buyer v_1, for the remainder of the game, from the pair $(\bar{\mu}_k^t,\bar{\pi}_{k-1})$, the beliefs μ about the state either move *upwards* or to 0 and the belief about the buyer type π either moves *downward* or to 1. Moreover, note that conditional on μ never going to 0 and π never going to 1, the *belief path* will be as follows :

$$(\bar{\mu}_k^t,\bar{\pi}_{k-1}) \longrightarrow (\bar{\mu}_{k-1}^{t-1},\bar{\pi}_{k-2}) \longrightarrow,\ldots,\longrightarrow (\bar{\mu}_2^{t-k+2},\bar{\pi}_1) \longrightarrow (\bar{\mu}_1^{t-k+1},\bar{\pi}_0) \longrightarrow (\bar{\mu}_0^{t-k},\bar{\pi}_0) \tag{27}$$

Note that since $\bar{\pi}_l = 1-\sigma^l$, the probability of moving from $\bar{\pi}_{l+1}$ to $\bar{\pi}_l$ is equal to σ. Note that once the good is bought and if $t-j$ periods are left, the seller now gets a payoff equal to $t-j$ for the remainder of the game since we have multiplied the continuation values with the scalar $\frac{1}{p}$. Moreover, note that once beliefs about the match quality μ move to 0, the seller gets a payoff of 0 for the rest of the game since the buyer would never choose to

buy. As a culmination of all of these facts we obtain :

$$\begin{aligned}
\hat{V}^{t-1}(\bar{\mu}_k^t, \bar{\pi}_{k-1}) &= \sum_{j=1}^{k-1} \sigma^{j-1}(1-\sigma)\left[\frac{(t-j)\bar{\mu}_k^t}{\bar{\mu}_{k-j}^{t-j}}\right] + \sigma^{k-1}\left[\frac{(t-k)\bar{\mu}_k^t}{\bar{\mu}_0^{t-k}}\right] \\
&= \bar{\mu}_k^t\left(\sum_{j=1}^{k-1} \sigma^{j-1}(1-\sigma)\left[\frac{(t-j)}{\bar{\mu}_{k-j}^{t-j}}\right] + \sigma^{k-1}\left[\frac{(t-k)}{\bar{\mu}_0^{t-k}}\right]\right) \\
&= \bar{\mu}_k^t\left(\sum_{j=1}^{k-1} \sigma^{j-1}(1-\sigma)\left[(k-j)\frac{1}{\mu_1^*} + (t-k)\frac{1}{\mu_2^*}\right] + \sigma^{k-1}\left[\frac{(t-k)}{\bar{\mu}_0^{t-k}}\right]\right) \\
&= \bar{\mu}_k^t\left(\sum_{j=1}^{k-1} \sigma^{j-1}(1-\sigma)\left[(k-j)\frac{1}{\mu_1^*}\right] + \left[\frac{(t-k)}{\bar{\mu}_2^*}\right]\right) \\
&= \bar{\mu}_k^t\left(\left(\frac{1}{\mu_1^*}\right)\left[k - \frac{1-\sigma^k}{1-\sigma}\right] + \left[\frac{(t-k)}{\bar{\mu}_2^*}\right]\right) \qquad (28)
\end{aligned}$$

Now let us return to the equality in (26). Observing that $1 - \bar{\pi}_{k-1} = \sigma^{k-1}$ and rearranging terms in (26) yields :

$$\frac{\mu}{\bar{\mu}_k^t} V^t(\mu_k^t, \pi) = \mu\left[\left(\frac{\pi - 1}{\sigma^{k-1}}\right)\left[\frac{t}{\bar{\mu}_k^t} - \frac{\hat{V}^{t-1}(\bar{\mu}_k^t, \bar{\pi}_{k-1})}{\bar{\mu}_k^t}\right] + \frac{t}{\bar{\mu}_k^t}\right] \qquad (29)$$

Now consider the expression $\left[\frac{t}{\bar{\mu}_k^t} - \frac{\hat{V}^{t-1}(\bar{\mu}_k^t, \bar{\pi}_{k-1})}{\bar{\mu}_k^t}\right]$ within (29). Using 28, this is equal to

$$\begin{aligned}
\frac{t}{\bar{\mu}_k^t} - \frac{\hat{V}^{t-1}(\bar{\mu}_k^t, \bar{\pi}_{k-1})}{\bar{\mu}_k^t} &= k\left(\frac{1}{\mu_1^*}\right) + (t-k)\left(\frac{1}{\mu_2^*}\right) - \left(\left(\frac{1}{\mu_1^*}\right)\left[k - \frac{1-\sigma^k}{1-\sigma}\right] + \left[\frac{(t-k)}{\bar{\mu}_2^*}\right]\right) \\
&= \left(\frac{1}{\mu_1^*}\right)\left[\frac{1-\sigma^k}{1-\sigma}\right] \qquad (30)
\end{aligned}$$

Hence, we now obtain :

$$\frac{\mu}{\bar{\mu}_k^t} V^t(\mu_k^t, \pi) = \mu\left[\left(\frac{\pi - 1}{\sigma^{k-1}}\right)\left(\frac{1}{\mu_1^*}\right)\left[\frac{1-\sigma^k}{1-\sigma}\right] + \frac{t}{\bar{\mu}_k^t}\right] \qquad (31)$$

Notice now we also obtain the value of $V^t(\mu_k^t, \pi)$:

$$V^t(\mu_k^t, \pi) = \left[\left(\frac{\pi - 1}{\sigma^{k-1}}\right)\left(\frac{\bar{\mu}_k^t}{\mu_1^*}\right)\left[\frac{1-\sigma^k}{1-\sigma}\right] + t\right] \qquad (32)$$

Note substituting $k = 0$, yields $V^t(\mu_0^t, \pi) = t$ which is indeed the case since both buyer types would play b at $\mu_0^t = \mu_2^*$ (no further information being given thereafter). Hence, the (32) may be used for computing $V^t(\mu_k^t, \pi)$ for any $0 \le k \le t$, and $\pi \ge \bar{\pi}_{k-1}$ where we define $\bar{\pi}_{-1} := 0 = \bar{\pi}_0$. We are now prepared to derive the value function $V^t(\mu, \pi)$

Deriving the closed form of $V^t(\mu, \pi)$: This section will be devoted to the derivation of the seller's value function $V^t(\mu, \pi)$. Let us fix $0 < t \le T$ and we shall define $V^t(., \pi)$ for each $\pi \in [0, 1]$ by partitioning the unit interval as $[0, 1] = \left(\bigcup_{l=0}^{t-1} [\bar{\pi}_l, \bar{\pi}_{l+1})\right) \cup [\bar{\pi}_t, 1) \cup \{1\}$ and defining $V^t(., \pi)$ for π in a given partition block

Consider $0 \le l < t$ and suppose $\pi \in [\bar{\pi}_l, \bar{\pi}_{l+1})$. We have the following cases :

1. *Case 1* : Suppose $\mu < \bar{\mu}_{l+1}^t$. Since buyer v_1 plays nb with probability one at μ, using the seller's action at $(\mu, \pi, t-1)$ we get the following :

$$\begin{aligned} V^t(\mu, \pi) &= \hat{V}^{t-1}(\mu, \pi) \\ &= \frac{\mu}{\bar{\mu}_l^{t-1}} V^{t-1}(\bar{\mu}_l^{t-1}, \pi) \\ &= \mu\left[\left(\frac{\pi - 1}{\sigma^{l-1}}\right)\left(\frac{1}{\mu_1^*}\right)\left[\frac{1 - \sigma^l}{1 - \sigma}\right] + \frac{t-1}{\bar{\mu}_l^{t-1}}\right] \end{aligned} \tag{33}$$

   where the last equality followed from (31)

2. *Case 2* : Suppose $\mu \in [\bar{\mu}_{l+1-i}^t, \bar{\mu}_{l-i}^t)$ where $0 \le i \le l-1$. buyer v_1 at this μ plays nb with probability α_{l-i} which is defined such that $\pi \longrightarrow_{\alpha_{l-i}} \bar{\pi}_{l-i}$. Note that $\bar{\mu}_{l+1-i}^t < \bar{\mu}_{l-i}^{t-1} < \bar{\mu}_{l-i}^t < \bar{\mu}_{l-i-1}^{t-1}$. We derive the value function by considering two sub-cases :

   (a) *Sub-case 1* : $\mu \in [\bar{\mu}_{l+1-i}^t, \bar{\mu}_{l-i}^{t-1})$. In this case from the seller's strategy in section 4.2.1 recall that the seller chooses the randomization $< z'(\mu, l-i, t-1) : (0, \bar{\mu}_{l-i-1}^{t-1}); 1 - z'(\mu, l-i, t-1) : (0, \bar{\mu}_{l-i}^{t-1}) >$. Hence, at $(\mu, \bar{\pi}_{l-i}, t-1)$, the seller gets a continuation value $\frac{\mu}{\bar{\mu}_{l-i-1}^{t-1}} V^{t-1}(\bar{\mu}_{l-i-1}^{t-1}, \pi)$ from choosing $(0, \bar{\mu}_{l-i-1}^{t-1})$ and a continuation value $\frac{\mu}{\bar{\mu}_{l-i}^{t-1}} V^{t-1}(\bar{\mu}_{l-i}^{t-1}, \pi)$ from choosing $(0, \bar{\mu}_{l-i}^{t-1})$. Since the seller is randomizing, it must be the case that these two values are the same. Let us first

confirm that this is indeed the case. Using (31), we show :

$$\begin{aligned}
\left(\frac{\bar{\pi}_{l-i}-1}{\sigma^{l-i-1}}\right)\left(\frac{1}{\mu_1^*}\right)\left[\frac{1-\sigma^{l-i}}{1-\sigma}\right]+\frac{t-1}{\bar{\mu}_{l-i}^{t-1}} &= \left(\frac{\bar{\pi}_{l-i}-1}{\sigma^{l-i-2}}\right)\left(\frac{1}{\mu_1^*}\right)\left[\frac{1-\sigma^{l-i-1}}{1-\sigma}\right]+\frac{t-1}{\bar{\mu}_{l-i-1}^{t-1}} \\
\left(\frac{\bar{\pi}_{l-i}-1}{\sigma^{l-i-1}}\right)\left(\frac{1}{\mu_1^*}\right)\left[1+\sigma+...+\sigma^{l-i-1}\right] &= \left(\frac{\bar{\pi}_{l-i}-1}{\sigma^{l-i-2}}\right)\left(\frac{1}{\mu_1^*}\right)\left[1+\sigma+...+\sigma^{l-i-2}\right]+\left(\frac{1}{\mu_2^*}-\frac{1}{\mu_1^*}\right) \\
\left(\frac{\bar{\pi}_{l-i}-1}{\sigma^{l-i-1}}\right)\left(\frac{1}{\mu_1^*}\right) &= \left(\frac{1}{\mu_2^*}-\frac{1}{\mu_1^*}\right) \\
\bar{\pi}_{l-i} &= 1-\sigma^{l-i}
\end{aligned}$$

where the last equality is indeed true by definition. Hence, the value function $V^t(\mu,\pi)$ is given by :

$$\begin{aligned}
V^t(\mu,\pi) &= \pi(1-\alpha_{l-i})[t]+((1-\pi)+\pi\alpha_{l-i})V^{t-t}(\mu,\bar{\pi}_{l-i}) \\
&= \left(\frac{\pi-\bar{\pi}_{l-i}}{1-\bar{\pi}_{l-i}}\right)[t]+\mu\left(\frac{1-\pi}{1-\bar{\pi}_{l-i}}\right)\left[\left(\frac{\bar{\pi}_{l-i}-1}{\sigma^{l-i-2}}\right)\left(\frac{1}{\mu_1^*}\right)\left[\frac{1-\sigma^{l-i-1}}{1-\sigma}\right]+\frac{t-1}{\bar{\mu}_{l-i-1}^{t-1}}\right] \quad (34)
\end{aligned}$$

(b) *Sub-case 2* : $\mu \in [\bar{\mu}_{l-i}^{t-1}, \bar{\mu}_{l-i}^{t})$, in this case the seller follows the randomization $< z(\mu, l-i, t-1) : (0, \bar{\mu}_{l-i-1}^{t-1}); 1-z(\mu, l-i, t-1) : (\bar{\mu}_{l-i}^{t-1}, \bar{\mu}_{l-i-1}^{t-1}) >$. Now from sub-case 1, we know that $\dfrac{V^{t-1}(\bar{\mu}_{l-i-1}^{t-1},\pi)}{\bar{\mu}_{l-i-1}^{t-1}} = \dfrac{V^{t-1}(\bar{\mu}_{l-i}^{t-1},\pi)}{\bar{\mu}_{l-i}^{t-1}}$, which turn implies that

$$\frac{\mu}{\bar{\mu}_{l-i-1}^{t-1}}V^{t-1}(\bar{\mu}_{l-i-1}^{t-1},\pi) = \left(\frac{\mu-\bar{\mu}_{l-i}^{t-1}}{\bar{\mu}_{l-i-1}^{t-1}-\bar{\mu}_{l-i}^{t-1}}\right)V^{t-1}(\bar{\mu}_{l-i-1}^{t-1},\pi)+\left(\frac{\bar{\mu}_{l-i-1}^{t-1}-\mu}{\bar{\mu}_{l-i-1}^{t-1}-\bar{\mu}_{l-i}^{t-1}}\right)V^{t-1}(\bar{\mu}_{l-i}^{t-1},\pi)$$

this implies that the experiments $(0, \bar{\mu}_{l-i-1}^{t-1})$ and $(\bar{\mu}_{l-i}^{t-1}, \bar{\mu}_{l-i-1}^{t-1})$ give the same continuation value at $(\mu, \bar{\pi}_{l-i}, t-1)$. As result the value function $V^t(\mu,\pi)$ is again given by :

$$V^t(\mu,\pi) = \left(\frac{\pi-\bar{\pi}_{l-i}}{1-\bar{\pi}_{l-i}}\right)[t]+\mu\left(\frac{1-\pi}{1-\bar{\pi}_{l-i}}\right)\left[\left(\frac{\bar{\pi}_{l-i}-1}{\sigma^{l-i-2}}\right)\left(\frac{1}{\mu_1^*}\right)\left[\frac{1-\sigma^{l-i-1}}{1-\sigma}\right]+\frac{t-1}{\bar{\mu}_{l-i-1}^{t-1}}\right] \quad (35)$$

3. *Case 3* : $\mu \in [\bar{\mu}_1^t, \bar{\mu}_0^t)$: Notice in this region buyer v_1 plays b but buyer v_2 plays nb. Hence, the seller learns the buyer's type. As a result,

$$V^t(\mu,\pi) = \pi[t]+\mu(1-\pi)\left(\frac{t-1}{\bar{\mu}_0^t}\right) \quad (36)$$

4. *Case 4* : $\mu \in [\bar{\mu}_0^t, 1]$: In this case, we simply have :

$$V^t(\mu, \pi) = t \tag{37}$$

Hence, to summarize all the cases, we have derived the following closed form of the value function $V^t(\mu, \pi)$:

$$V^t(\mu,\pi) = \begin{cases} t & \text{if } \mu \in [\bar{\mu}_0^t, 1] \\ \pi[t] + \mu(1-\pi)\left(\dfrac{t-1}{\bar{\mu}_0^t}\right) & \text{if } \mu \in [\bar{\mu}_1^t, \bar{\mu}_0^t) \\ \left(\dfrac{\pi - \bar{\pi}_{l-i}}{1-\bar{\pi}_{l-i}}\right)[t] + \mu\left(\dfrac{1-\pi}{1-\bar{\pi}_{l-i}}\right)\left[\left(\dfrac{\bar{\pi}_{l-i}-1}{\sigma^{l-i-2}}\right)\left(\dfrac{1}{\mu_1^*}\right)\left[\dfrac{1-\sigma^{l-i-1}}{1-\sigma}\right] + \dfrac{t-1}{\bar{\mu}_{l-i-1}^{t-1}}\right] & \text{if } \mu \in [\bar{\mu}_{l+1-i}^t, \bar{\mu}_{l-i}^t) \\ \mu\left[\left(\dfrac{\pi-1}{\sigma^{l-1}}\right)\left(\dfrac{1}{\mu_1^*}\right)\left[\dfrac{1-\sigma^l}{1-\sigma}\right] + \dfrac{t-1}{\bar{\mu}_l^{t-1}}\right] & \text{if } \mu < \bar{\mu}_{l+1}^t \end{cases} \tag{38}$$

where $0 \le i \le l-1$. Now consider $\pi \in [\bar{\pi}_t, 1)$. One can show that the closed form representation for $V^t(\mu, \pi)$ for this case can be found simply by substituting $l = t-1$ in the closed form provided above. Hence, the "expression" for the value function is the same for all $\pi \in [\bar{\pi}_{t-1}, 1)$. However, since the values of π differ on $[\bar{\pi}_{t-1}, \bar{\pi}_t)$ and $[\bar{\pi}_t, 1)$, the ranking of experiments on $\Gamma(\mu)$ given by $\succsim_{(\mu,\pi,t)}$ would change depending on which region π lies in. The case $\pi = 1$ is straightforward :

$$V^t(\mu,\pi) = \begin{cases} t & \text{if } \mu \ge \mu_1^* \\ \dfrac{\mu}{\mu_1^*}(t-1) & \text{if } \mu < \mu_1^* \end{cases}$$

Establishing Properties : We shall show that the value function defined in the above section satisfies properties 1-6 in Claim 3. Fix $0 < t \le T$ and consider $0 \le l < t$ and $\pi \in [\bar{\pi}_l, \bar{\pi}_{l+1})$. We shall show that $V^t(., \pi)$ satisfies conditions 1-6.

Corresponding to the set of points $\{\bar{x}_i\}_{1 \le i \le n} \subseteq [0,1]$ in the hypothesis of Claim 3, we shall, in the present context, define these points to be $\{\bar{\mu}_{l+1-i}^t\}_{0 \le i \le l+1} \subseteq [0,1]$. Hence, in this case, $n = l+2$. We now establish the conditions for $V^t(., \pi)$:

1. (*Zero at zero*) : Since $V^t(., \pi)$ is in fact linear on $[0, \bar{\mu}_{l+1}^t)$, we get :

$$V^t(0, \pi) = 0 \tag{39}$$

2. (*Constant at the end*) : From (38) it clear that for all $\mu \in [\bar{\mu}_0^t, 1]$

$$V^t(\mu, \pi) = t \tag{40}$$

3. (*Piecewise Affine*) : From (38) and the derivation of (31), it is clear $V^t(0, \pi)$ is piecewise positively affine. For $\mu \in [\bar{\mu}_{l+1-i}^t, \bar{\mu}_{l-i}^t)$, we have :

$$V^t(\mu, \pi) = a_i \mu + b_i \tag{41}$$

where $a_i := \left(\frac{1-\pi}{1-\bar{\pi}_{l-i}}\right)\left[\left(\frac{\bar{\pi}_{l-i}-1}{\sigma^{l-i-2}}\right)\left(\frac{1}{\mu_1^*}\right)\left[\frac{1-\sigma^{l-i-1}}{1-\sigma}\right] + \frac{t-1}{\bar{\mu}_{l-i-1}^{t-1}}\right] \geq 0$ and $b_i := \left(\frac{\pi - \bar{\pi}_{l-i}}{1-\bar{\pi}_{l-i}}\right)[t]$.

For the regions $[0, \bar{\mu}_{l+1}^t)$ and $[\bar{\mu}_1^t, \bar{\mu}_0^t)$, it is clear that $V^t(., \pi)$ is postively linear and positively affine respectively.

4. (*Jumps*) : We will now show that for $\hat{\mu}_{l+1}^t$ and $0 \leq j \leq l$

$$V^t(\bar{\mu}_{l+1}^t, \pi) \geq \lim_{\mu' \uparrow \bar{\mu}_{l+1}^t} V^t(\mu', \pi) \tag{42}$$

$$V^t(\bar{\mu}_j^t, \pi) > \lim_{\mu' \uparrow \bar{\mu}_j^t} V^t(\mu', \pi) \tag{43}$$

(a) We first show (42). From (38) we know that :

$$\lim_{\mu' \uparrow \bar{\mu}_{l+1}^t} V^t(\mu', \pi) = \bar{\mu}_{l+1}^t \left[\left(\frac{\pi-1}{\sigma^{l-1}}\right)\left(\frac{1}{\mu_1^*}\right)\left[\frac{1-\sigma^l}{1-\sigma}\right] + \frac{t-1}{\bar{\mu}_l^{t-1}}\right]$$

$$V^t(\bar{\mu}_{l+1}^t, \pi) = \left(\frac{\pi - \bar{\pi}_l}{1-\bar{\pi}_l}\right)[t] + \bar{\mu}_{l+1}^t \left(\frac{1-\pi}{1-\bar{\pi}_l}\right)\left[\left(\frac{\bar{\pi}_l - 1}{\sigma^{l-2}}\right)\left(\frac{1}{\mu_1^*}\right)\left[\frac{1-\sigma^{l-1}}{1-\sigma}\right] + \frac{t-1}{\bar{\mu}_{l-1}^{t-1}}\right] \tag{44}$$

From arguments analogous to those presented in Case 2 part a) in deriving the value function above, we have :

$$\left(\frac{\bar{\pi}_l - 1}{\sigma^{l-2}}\right)\left(\frac{1}{\mu_1^*}\right)\left[\frac{1-\sigma^{l-1}}{1-\sigma}\right] + \frac{t-1}{\bar{\mu}_{l-1}^{t-1}} = \left(\frac{\bar{\pi}_l - 1}{\sigma^{l-1}}\right)\left(\frac{1}{\mu_1^*}\right)\left[\frac{1-\sigma^l}{1-\sigma}\right] + \frac{t-1}{\bar{\mu}_l^{t-1}}$$

Hence, (44) now becomes

$$
\begin{aligned}
V^t(\bar{\mu}^t_{l+1},\pi) &= \left(\frac{\pi-\bar{\pi}_l}{1-\bar{\pi}_l}\right)[t] + \bar{\mu}^t_{l+1}\left(\frac{1-\pi}{1-\bar{\pi}_l}\right)\left[\left(\frac{\bar{\pi}_l-1}{\sigma^{l-1}}\right)\left(\frac{1}{\mu^*_1}\right)\left[\frac{1-\sigma^l}{1-\sigma}\right]+\frac{t-1}{\bar{\mu}^{t-1}_l}\right] \\
&= \left(\frac{\pi-\bar{\pi}_l}{1-\bar{\pi}_l}\right)[t] + \bar{\mu}^t_{l+1}\left[\left(\frac{\pi-1}{\sigma^{l-1}}\right)\left(\frac{1}{\mu^*_1}\right)\left[\frac{1-\sigma^l}{1-\sigma}\right]+\left(\frac{1-\pi}{1-\bar{\pi}_l}\right)\left(\frac{t-1}{\bar{\mu}^{t-1}_l}\right)\right] \\
&= \left(\frac{\pi-\bar{\pi}_l}{1-\bar{\pi}_l}\right)\left[t-\bar{\mu}^t_{l+1}\left(\frac{t-1}{\bar{\mu}^{t-1}_l}\right)\right] + \bar{\mu}^t_{l+1}\left[\left(\frac{\pi-1}{\sigma^{l-1}}\right)\left(\frac{1}{\mu^*_1}\right)\left[\frac{1-\sigma^l}{1-\sigma}\right]+\left(\frac{t-1}{\bar{\mu}^{t-1}_l}\right)\right] \\
&\geq \lim_{\mu'\uparrow\bar{\mu}^t_{l+1}} V^t(\mu',\pi)
\end{aligned}
$$

The last inequality follows from the fact that $\bar{\mu}^t_{l+1} < \bar{\mu}^{t-1}_l$. Moreover, the inequality would be strict if and only if $\pi > \bar{\pi}_l$.

(b) We now establish (43) for $2 \leq j \leq l$ and shall treat the cases $j = 0$ and $j = 1$ seperately. Notice this is the equivalent to establishing

$$
V^t(\bar{\mu}^t_{l+1-i},\pi) > \lim_{\mu'\uparrow\bar{\mu}^t_{l+1-i}} V^t(\mu',\pi) \tag{45}
$$

for $1 \leq i \leq l-1$. Note from (38) and re-arranging terms as above yields :

$$
\begin{aligned}
\lim_{\mu'\uparrow\bar{\mu}^t_{l+1-i}} V^t(\mu',\pi) &= \left(\frac{\pi-\bar{\pi}_{l+1-i}}{1-\bar{\pi}_{l+1-i}}\right)[t] + \bar{\mu}^t_{l+1-i}\left(\frac{1-\pi}{1-\bar{\pi}_{l+1-i}}\right)\left[\left(\frac{\bar{\pi}_{l+1-i}-1}{\sigma^{l-i-1}}\right)\left(\frac{1}{\mu^*_1}\right)\left[\frac{1-\sigma^{l-i}}{1-\sigma}\right]+\frac{t-1}{\bar{\mu}^{t-1}_{l-1}}\right] \\
&= \left(\frac{\pi-\bar{\pi}_{l+1-i}}{1-\bar{\pi}_{l+1-i}}\right)\left[t-\frac{\bar{\mu}^t_{l+1-i}(t-1)}{\bar{\mu}^{t-1}_{l-1}}\right] + \bar{\mu}^t_{l+1-i}\left[\left(\frac{\pi-1}{\sigma^{l-i-1}}\right)\left(\frac{1}{\mu^*_1}\right)\left[\frac{1-\sigma^{l-i}}{1-\sigma}\right]+\frac{t-1}{\bar{\mu}^{t-1}_{l-1}}\right]
\end{aligned}
$$

Again, similar to part a) and the re-arrangement above, we have :

$$
V^t(\bar{\mu}^t_{l+1-i},\pi) = \left(\frac{\pi-\bar{\pi}_{l-i}}{1-\bar{\pi}_{l-i}}\right)\left[t-\bar{\mu}^t_{l+1-i}\left(\frac{t-1}{\bar{\mu}^{t-1}_{l-i}}\right)\right] + \bar{\mu}^t_{l+1-i}\left[\left(\frac{\pi-1}{\sigma^{l-i-1}}\right)\left(\frac{1}{\mu^*_1}\right)\left[\frac{1-\sigma^{l-i}}{1-\sigma}\right]+\left(\frac{t-1}{\bar{\mu}^{t-1}_{l-i}}\right)\right]
$$

Since $\bar{\pi}_{l+1-i} > \bar{\pi}_{l-i}$, it follows that $V^t(\bar{\mu}^t_{l+1-i},\pi) > \lim_{\mu'\uparrow\bar{\mu}^t_{l+1-i}} V^t(\mu',\pi)$

(c) Consider $j = 1$. We get that :

$$\begin{aligned}
\lim_{\mu' \uparrow \bar{\mu}_1^t} V^t(\mu', \pi) &= \left(\frac{\pi - \bar{\pi}_1}{1 - \bar{\pi}_1}\right)\left[t - \frac{\bar{\mu}_1^t(t-1)}{\bar{\mu}_0^{t-1}}\right] + \frac{\bar{\mu}_1^t(t-1)}{\bar{\mu}_0^{t-1}} \\
&< \pi\left[t - \frac{\bar{\mu}_1^t(t-1)}{\bar{\mu}_0^{t-1}}\right] + \frac{\bar{\mu}_1^t(t-1)}{\bar{\mu}_0^{t-1}} \\
&= V^t(\bar{\mu}_1^t, \pi)
\end{aligned}$$

(d) Consider $j = 0$. Since $\pi[t] + (1-\pi)(t-1) < t$, the conclusion follows.

5. (*Decreasing averages for $j \leq l$*). We now wish to show for $0 \leq i \leq l-1$

$$\frac{V^t(\bar{\mu}_{l+1}^t, \pi)}{\bar{\mu}_{l+1}^t} < \frac{V^t(\bar{\mu}_l^t, \pi)}{\bar{\mu}_l^t} \tag{46}$$

$$\frac{V^t(\bar{\mu}_{l-i}^t, \pi)}{\bar{\mu}_{l-i}^t} \geq \frac{V^t(\bar{\mu}_{l-i-1}^t, \pi)}{\bar{\mu}_{l-i-1}^t} \tag{47}$$

(a) We establish (46) by following the chain of inequalities below :

$$\begin{aligned}
\frac{V^t(\bar{\mu}_{l+1}^t, \pi)}{\bar{\mu}_{l+1}^t} &< \frac{V^t(\bar{\mu}_l^t, \pi)}{\bar{\mu}_l^t} \\
\left(\frac{\pi - 1}{\sigma^l}\right)\left(\frac{1}{\mu_1^*}\right)\left[\frac{1 - \sigma^{l+1}}{1 - \sigma}\right] + \frac{t}{\bar{\mu}_l^t} &< \left(\frac{\pi - 1}{\sigma^{l-1}}\right)\left(\frac{1}{\mu_1^*}\right)\left[\frac{1 - \sigma^l}{1 - \sigma}\right] + \frac{t}{\bar{\mu}_l^t} \\
\left(\pi - 1\right)\left(\frac{1}{\mu_1^*}\right)\left[\frac{1}{\sigma^l}\left[\frac{1 - \sigma^{l+1}}{1 - \sigma}\right] - \frac{1}{\sigma^{l-1}}\left[\frac{1 - \sigma^l}{1 - \sigma}\right]\right] &< \frac{1}{\mu_2^*} - \frac{1}{\mu_1^*} \\
\left(\pi - 1\right)\left[\frac{1}{\sigma^l}\right] &< -\sigma \\
\pi &< 1 - \sigma^{l+1}
\end{aligned}$$

where the last strictly inequality follows from hypothesis since we consider $\pi \in [\bar{\pi}_l, \bar{\pi}_{l+1})$

(b) We now establish (47) by following the chain of inequalities below :

$$\begin{aligned}
\frac{V^t(\bar{\mu}^t_{l-i},\pi)}{\bar{\mu}^t_{l-i}} &\geq \frac{V^t(\bar{\mu}^t_{l-i-1},\pi)}{\bar{\mu}^t_{l-i-1}} \\
\left(\frac{\pi-1}{\sigma^{l-i-1}}\right)\left(\frac{1}{\mu^*_1}\right)\left[\frac{1-\sigma^{l-i}}{1-\sigma}\right]+\frac{t}{\bar{\mu}^t_{l-i}} &\geq \left(\frac{\pi-1}{\sigma^{l-1-2}}\right)\left(\frac{1}{\mu^*_1}\right)\left[\frac{1-\sigma^{l-i-1}}{1-\sigma}\right]+\frac{t}{\bar{\mu}^t_{l-i-1}} \\
\left(\pi-1\right)\left(\frac{1}{\mu^*_1}\right)\left[\frac{1}{\sigma^{l-i-1}}\left[\frac{1-\sigma^{l-i}}{1-\sigma}\right]-\frac{1}{\sigma^{l-1-2}}\left[\frac{1-\sigma^{l-i-1}}{1-\sigma}\right]\right] &\geq \frac{1}{\mu^*_2}-\frac{1}{\mu^*_1} \\
\left(\pi-1\right)\left[\frac{1}{\sigma^{l-i-1}}\right] &\geq -\sigma \\
\pi &\geq 1-\sigma^{l-i}
\end{aligned}$$

The last inequality follows from the fact that $\pi \geq \bar{\pi}_l \geq \bar{\pi}_{l-i}$. The inequality must necessarily be strict if $i \neq 0$ or $\pi > \bar{\pi}_l$. For $i = 0$ and $\pi = \bar{\pi}_l$ implies the above will infact be an equality.

6. (*Decreasing slopes for $j \leq l$*) We wish to show for all $2 \leq j \leq l$:

$$\frac{V^t(\bar{\mu}^t_{j-1},\pi)-V^t(\bar{\mu}^t_j,\pi)}{\bar{\mu}^t_{j-1}-\bar{\mu}^t_j} \geq \frac{V^t(\bar{\mu}^t_{j-2},\pi)-V^t(\bar{\mu}^t_{j-1},\pi)}{\bar{\mu}^t_{j-2}-\bar{\mu}^t_{j-1}} \tag{48}$$

For each $1 \leq j \leq l$ define :

$$\Delta(j) := \frac{V^t(\bar{\mu}^t_{j-1},\pi)-V^t(\bar{\mu}^t_j,\pi)}{\bar{\mu}^t_{j-1}-\bar{\mu}^t_j} \tag{49}$$

Hence, it suffices to show that $\Delta(j) \geq \Delta(j-1)$ for all $2 \leq j \leq l$. First we derive a closed-form for (49) :

$$\begin{aligned}
\bar{\mu}^t_{j-1}-\bar{\mu}^t_j &= \left(\frac{t}{(j-1)\left(\frac{1}{\mu^*_1}\right)+(t-j+1)\left(\frac{1}{\mu^*_2}\right)}\right)-\left(\frac{t}{(j)\left(\frac{1}{\mu^*_1}\right)+(t-j)\left(\frac{1}{\mu^*_2}\right)}\right) \\
&= \frac{1}{t}\left(\frac{1}{\mu^*_1}-\frac{1}{\mu^*_2}\right)\bar{\mu}^t_{j-1}\bar{\mu}^t_j \\
&= \frac{1}{t}\left(\frac{1}{\mu^*_1}\right)\sigma\bar{\mu}^t_{j-1}\bar{\mu}^t_j
\end{aligned}$$

Hence,

$$
\begin{aligned}
\Delta(j) &= \frac{\left(\pi-1\right)\left(\frac{1}{\mu_1^*}\right)\left[\frac{1}{\sigma^{j-2}}\left[\frac{1-\sigma^{j-1}}{1-\sigma}\right]\bar{\mu}_{j-1}^t-\frac{1}{\sigma^{j-1}}\left[\frac{1-\sigma^{j}}{1-\sigma}\right]\bar{\mu}_{j}^t\right]}{\frac{1}{t}\left(\frac{1}{\mu_1^*}\right)\sigma\bar{\mu}_{j-1}^t\bar{\mu}_j^t} \\
&= \left(\frac{1-\pi}{\sigma}\right)\left[\frac{1}{\sigma^{j-1}}\left[\frac{1-\sigma^{j}}{1-\sigma}\right]\left(\frac{t}{\bar{\mu}_{j-1}^t}\right)-\frac{1}{\sigma^{j-2}}\left[\frac{1-\sigma^{j-1}}{1-\sigma}\right]\left(\frac{t}{\bar{\mu}_{j}^t}\right)\right] \\
&= \left(\frac{1-\pi}{1-\sigma}\right)\left[\left[\frac{1}{\sigma^{j}}-1\right]\left(\frac{t}{\bar{\mu}_{j-1}^t}\right)-\left[\frac{1}{\sigma^{j-1}}-1\right]\left(\frac{t}{\bar{\mu}_{j}^t}\right)\right] \\
&= \left(\frac{1-\pi}{1-\sigma}\right)\left[\left(\frac{1}{\sigma^{j}}\right)\left[j\left(\frac{1-\sigma}{\mu_1^*}\right)+(t-j)\left(\frac{1-\sigma}{\mu_2^*}\right)+\left(\frac{1}{\mu_2^*}-\frac{1}{\mu_1^*}\right)\right]+\left(\frac{1}{\mu_1^*}-\frac{1}{\mu_2^*}\right)\right] \\
&= \left(\frac{1-\pi}{1-\sigma}\right)\left[\tilde{\Delta}(j)+\left(\frac{1}{\mu_1^*}-\frac{1}{\mu_2^*}\right)\right]
\end{aligned}
$$

where $\tilde{\Delta}(j) := \left(\frac{1}{\sigma^j}\right)\left[j\left(\frac{1-\sigma}{\mu_1^*}\right)+(t-j)\left(\frac{1-\sigma}{\mu_2^*}\right)+\left(\frac{1}{\mu_2^*}-\frac{1}{\mu_1^*}\right)\right]$. Hence, since $\frac{1-\pi}{1-\sigma}>0$, it suffices to show that $\tilde{\Delta}(j)\geq\tilde{\Delta}(j-1)$. Now using the fact that $\mu_1^*=(1-\sigma)\mu_2^*$, we have that :

$$
\tilde{\Delta}(j) := \left(\frac{1}{\sigma^j\mu_2^*}\right)\left[j+(t-j)(1-\sigma)-\frac{\sigma}{1-\sigma}\right]
$$

Now,

$$
\begin{aligned}
\tilde{\Delta}(j)-\tilde{\Delta}(j-1) &= \frac{1}{\sigma^{j-1}\mu_2^*}\left[\left(\frac{1}{\sigma}\right)\left[j+(t-j)(1-\sigma)-\frac{\sigma}{1-\sigma}\right]-\left[(j-1)+(t-j+1)(1-\sigma)-\frac{\sigma}{1-\sigma}\right]\right] \\
&= \frac{1}{\sigma^{j-1}\mu_2^*}\left[\left(\frac{1}{\sigma}\right)\left[j+(t-j)(1-\sigma)-\frac{\sigma}{1-\sigma}\right]-\left[(j-1)+(t-j+1)(1-\sigma)-\frac{\sigma}{1-\sigma}\right]\right] \\
&\geq \frac{1}{\sigma^{j-1}\mu_2^*}\left[\left(\frac{1}{\sigma}\right)\left[j+(t-j)(1-\sigma)\right]-\left(\frac{1}{\sigma}\right)\left[(j-1)+(t-j+1)(1-\sigma)\right]-1\right] \\
&= 0
\end{aligned}
$$

Hence, we have established the claim. Now the conclusions follows straightforwardly. The set of break points in 1 arise from the jumps lemma above. Consider now 2. Part (a) follows directly from part (b) of Claim 3 since $\mu<\bar{\mu}_l^t$ and the fact that the value of the concavification of $V^t(.,\pi)$ at μ equals the value generated by

the optimal experiment.Part (b) follows from the decreasing averages result above and (c) follows again from case (b) of the decreasing averages result above wherein we argued that the inequality presented in the case of $i = 0$ and $\pi = \bar{\pi}_l$ will infact be an equality. Consider now 3. Part (a) follows from conclusion (c) in Claim 3 which relies on the decreasing slopes property. Lastly, Part (b) follows as a corollary of 2 (c).

7.3 Proof of Proposition 4

By arguments completely analogous to those presented in the proof of Proposition 5, we can derive the value function to be :

$$V^t(\mu,\pi) = \begin{cases} t & \text{if } \mu \in [\bar{\mu}_0^t, 1] \\ \pi[t] + \mu(1-\pi)\left(\dfrac{t-1}{\bar{\mu}_0^t}\right) & \text{if } \mu \in [\bar{\mu}_1^t, \bar{\mu}_0^t) \\ \left(\dfrac{\pi - \bar{\pi}_{t-i}}{1-\bar{\pi}_{t-i}}\right)[t] + \mu\left(\dfrac{1-\pi}{1-\bar{\pi}_{t-i}}\right)\left[\left(\dfrac{\bar{\pi}_{t-i}-1}{\sigma^{t-i-2}}\right)\left(\dfrac{1}{\mu_1^*}\right)\left[\dfrac{1-\sigma^{t-i-1}}{1-\sigma}\right] + \dfrac{t-1}{\bar{\mu}_{t-i-1}^{t-1}}\right] & \text{if } \mu \in [\bar{\mu}_{t+1-i}^t, \bar{\mu}_{t-i}^t) \\ \mu\left[\left(\dfrac{\pi-1}{\sigma^{t-2}}\right)\left(\dfrac{1}{\mu_1^*}\right)\left[\dfrac{1-\sigma^{t-1}}{1-\sigma}\right] + \dfrac{t-1}{\bar{\mu}_{t-1}^{t-1}}\right] & \text{if } \mu < \bar{\mu}_t^t \end{cases} \tag{50}$$

where $1 \le i \le t-1$. Note that the expression for this value function (as we remarked in the proof of Proposition 8) is exactly equal to the expression in (38) if we substitute $l = t-1$ and consider the range $0 \le i \le l-1 = t-2$. The difference only lies in the value of π. In this case we have $\pi \in [\bar{\pi}_t, 1)$ instead of $\pi \in [\bar{\pi}_{t-1}, \bar{\pi}_t)$. This change in the value of π preserves properties 1-4 and 6 of claim 3. The only alteration that occurs is in the average values in condition 5 in that they hold now for all $i \ge 1$ i.e in this context we now get for $0 \le i \le t-1$

$$\frac{V^t(\bar{\mu}_{t-i}^t, \pi)}{\bar{\mu}_{t-i}^t} \ge \frac{V^t(\bar{\mu}_{t-i-1}^t, \pi)}{\bar{\mu}_{t-i-1}^t}$$

The implication that follows is that $(0, \bar{\mu}_t)$ now becomes the optimal experiment when $\mu < \bar{\mu}_t^t$ and for $\mu \in [\bar{\mu}_{t-i}^t, \bar{\mu}_{t-i-1}^t)$, the optimal experiment becomes $(\bar{\mu}_{t-i}^t, \bar{\mu}_{t-i-1}^t)$. The rest of the conclusions follows due to exactly the same reasons as Proposition 5.

7.4 Proof of Proposition 5

We prove each of the statements below :

1. The optimality of the seller's strategy is already implied by Propositions 4 and 5.

We need only establish optimality of the buyer's strategy. Consider first buyer v_2. Optimality of buyer 2 actions will be follow directly Claim 1. Note that the seller's strategy is such that posteriors never exceed μ_2^*. If $\mu < \mu_2^*$, we know that future posteriors will be less than or equal to μ_2^* and hence that buyer v_2 strictly prefers nb to b as prescribed by the strategy. If $\mu \geq \mu_2^*$ note that the seller gives no further information and hence, it is optimal to play b.

Now consider buyer v_1. Consider any stage (μ, π, t) such that $\pi = 1$. If $\mu < \mu_1^*$, then it again follows from Claim 1 that buyer v_1 strictly prefers nb to b. If $\mu \geq \mu_1^*$, then no further information is provided and v_1 optimally plays b.

Now consider a stage where the state of the game is (μ, π, t) such that $\pi \in [\bar{\pi}_l, \bar{\pi}_{l+1})$ where $0 \leq l < t$. We confirm the optimality of buyer v_1's strategy by considering the following cases :

(a) *Case 1* : Suppose $0 < \mu < \bar{\mu}_{l+1}^t$. Consider first the case $\mu < \mu_1^*$. From Claim 1 it follows that v_1 strictly prefers nb to b as prescribed by the strategies. Now consider the case $\mu \in [\mu_1^*, \bar{\mu}_{l+1}^t)$. If the buyer has not bought, then the seller provides information $(0, \bar{\mu}_l^{t-1})$ and at $\bar{\mu}_l^{t-1}$, the buyer v_1 would be indifferent between buying and not buying. This gives rise to the following payoffs :

$$\begin{aligned} b &: (\mu v_1 - p)t \\ nb &: \frac{\mu}{\bar{\mu}_l^{t-1}}(\bar{\mu}_l^{t-1} v_1 - p)(t-1) \end{aligned} \tag{51}$$

It follows that nb is strictly better if and only if $\mu < \bar{\mu}_{l+1}^t$ which is true by hypothesis. Hence, the strategy prescription is optimal.

(b) *Case 2* : Suppose $\mu \in [\bar{\mu}_{l+1-i}^t, \bar{\mu}_{l-i}^t)$ where $0 \leq i \leq l-1$. Notice that $\bar{\mu}_{l+1-i}^t < \bar{\mu}_{l-i}^{t-1} < \bar{\mu}_{l-i}^t < \bar{\mu}_{l-i-1}^{t-1}$. Consider first the case $\mu \in [\bar{\mu}_{l+1-i}^t, \bar{\mu}_{l-i}^{t-1})$.If nb is played, the belief about the buyer's type moves to $\bar{\pi}_{l-i}$ and then the seller uses the randomisation $< z'(\mu, l-i, t-1) : (0, \bar{\mu}_{l-i-1}^{t-1}); 1-z'(\mu, l-i, t-1) : (0, \bar{\mu}_{l-i}^{t-1}) >$. This yields the following payoffs from actions to buyer v_1 :

$$\begin{aligned} b &: (\mu v_1 - p)t \\ nb &: z'(\mu, l-i, t-1)\frac{\mu}{\bar{\mu}_{l-i-1}^{t-1}}(\bar{\mu}_{l-i-1}^{t-1} v_1 - p)(t-1) + (1 - z'(\mu, l-i, t-1))\frac{\mu}{\bar{\mu}_{l-i}^{t-1}}(\bar{\mu}_{l-i}^{t-1} v_1 - p)(t-1) \end{aligned} \tag{52}$$

But now recall that $z'(\mu, l-i, t-1)$ was defined in the strategy specification precisely to equate the above two quantities. Hence the randomisation α_{l-i} is indeed optimal for buyer v_1. Consider now the case $\mu \in [\bar{\mu}^{t-1}_{l-i}, \bar{\mu}^{t}_{l-i})$. If nb is played, the belief about the buyer type moves to $\bar{\pi}_{l-i}$ and then the seller uses the randomisation $< z'(\mu, l-i, t-1) : (0, \bar{\mu}^{t-1}_{l-i-1}); 1 - z'(\mu, l-i, t-1) : (0, \bar{\mu}^{t-1}_{l-i}) >$. Again, by definition of $z'(\mu, l-i, t-1)$ it follows that b and nb give the same payoff making the randomization played by the buyer optimal

(c) *Case 3* : Suppose $\mu \geq \bar{\mu}^t_1$. First consider the case $\mu \geq \mu^*_2$. If nb is played, no further information will be provided and since μ is now strictly above buyer v_1's threshold, he would strictly prefer the action b over nb. Now consider $\mu \in [\bar{\mu}^t_1, \mu^*_2)$. Notice in this case, actions would perfectly reveal the type of the buyer. He believes it is buyer 2 if nb is played and provides $(0, \mu^*_2)$ in the next round. This leads to the following payoffs :

$$\begin{aligned} b &: (\mu v_1 - p)t \\ nb &: \frac{\mu}{\mu^*_2}(\mu^*_2 v_1 - p)(t-1) \end{aligned} \tag{53}$$

It follows that b is better thannb if and only if $\mu \geq \bar{\mu}^t_1$ which is true by hypothesis.

This establishes the optimality of buyer actions for the case $\pi \in [\bar{\pi}_l, \bar{\pi}_{l+1})$ and $0 \leq l < t$. The case $\pi \in [\bar{\pi}_t, 1)$ follows completely analogously. As a culmination of arguments for optimality of buyer's strategy presented above and arguments presented in the proofs of Propositions 4 and 5 for optimality of the seller's strategy, we have established that the strategies constructed indeed constitutes an equilibrium.

2. Notice since $\mu_0 < \mu^*_1$, and if $\pi_0 \in [\bar{\pi}_l, \bar{\pi}_{l+1})$, the experiment chosen in the first period is $(0, \bar{\mu}^t_l)$. If the posterior generated is $\bar{\mu}^t_l$ and nb is played, then the next period experiment is $(0, \bar{\mu}^{t-1}_{l-1})$. Again, if posterior $\bar{\mu}^{t-1}_{l-1}$ is generated and nb is played again, then experiment $(0, \bar{\mu}^{t-2}_{l-2})$ is chosen. If beliefs keep moving upwards as nb is keep getting chosen, then on-path only experiments of the form $\{(0, \bar{\mu}^{t-j}_{l-j})\}_{0 \leq j \leq l}$ are chosen till a point arrives when beliefs settle at $\bar{\mu}^{t-l}_0 = \mu^*_2$ and no further information is given by the seller. This is also true for the case $\pi_0 \in [\bar{\pi}_t, 1)$ where $(0, \mu^t_t) = (0, \mu^*_1)$ is chosen at the beginning and no further information is provided. Now consider any experiment $(0, \mu')$ centered at $0 < \mu < \mu'$. Since the experiment has a two point support only two messages $\{m_0, m_1\}$ are needed and a stochastic map $e \in \mathcal{E}$ that

generates it can be represented in the following form :

$$\begin{aligned} e(G) &= < m_0 : 1; m_1 : 0 > \\ e(B) &= < m_0 : \beta; m_1 : 1-\beta > \end{aligned} \tag{54}$$

where $\beta \in (0,1)$ such that $\dfrac{\mu}{\mu'} = \mu + (1-\mu)\beta$. But notice that any such experiment only garbles the bad state and truthfully reports the good state.

We now establish that the experiments chosen second period onwards are increasingly informative i.e we show that $(0, \bar{\mu}_{l-j-1}^{t-j-1})$ is more informative than $(0, \bar{\mu}_{l-j}^{t-j})$ in the Blackwell sense for $j \geq 1$. Now, let e_h be the stochastic map corresponding $(0, \bar{\mu}_{l-j-h}^{t-j-h})$. We will show that e_0 is a garbling of e_1. We know that each of these experiments only garbles the bad state. Let β^h be the probability under experiment e^h, the message m_0 is generated in state $\omega = B$. It suffices to show that $\beta_0 > \beta_1$.

Now $(0, \bar{\mu}_{l-j}^{t-j})$ is centered at $\bar{\mu}_{l-j+1}^{t-j+1}$ and $(0, \bar{\mu}_{l-j-1}^{t-j-1})$ is centered at $\bar{\mu}_{l-j}^{t-j}$. Letting $s = t-j$ and $k = l-j$ it follows that :

$$\beta_0 = \frac{\dfrac{1}{\bar{\mu}_k^s} - 1}{\dfrac{1}{\bar{\mu}_{k+1}^{s+1}} - 1} \quad ; \quad \beta_1 = \frac{\dfrac{1}{\bar{\mu}_{k-1}^{s-1}} - 1}{\dfrac{1}{\bar{\mu}_k^s} - 1}$$

Now it follows that :

$$\beta_0 > \beta_1 \iff \left(\frac{1}{\bar{\mu}_k^s} - 1\right)^2 > \left(\frac{1}{\bar{\mu}_{k-1}^{s-1}} - 1\right)\left(\frac{1}{\bar{\mu}_{k+1}^{s+1}} - 1\right)$$

which is equivalent to :

$$\left(\frac{k}{s}\left(\frac{1}{\mu_1^*}\right) + \left(1 - \frac{k}{s}\right)\left(\frac{1}{\mu_2^*}\right) - 1\right) > \sqrt{\left(\frac{k-1}{s-1}\left(\frac{1}{\mu_1^*}\right) + \left(1 - \frac{k-1}{s-1}\right)\left(\frac{1}{\mu_2^*}\right) - 1\right)\left(\frac{k+1}{s+1}\left(\frac{1}{\mu_1^*}\right) + \left(1 - \frac{k+1}{s+1}\right)\left(\frac{1}{\mu_2^*}\right) - 1\right)} \tag{55}$$

We argue that the above strict inequality holds true. Let us define the function $d : (-\infty, s) \to \mathbb{R}$ where $d(x) = \dfrac{k-x}{s-x}\left(\dfrac{1}{\mu_1^*}\right) + \left(1 - \dfrac{k-x}{s-x}\right)\left(\dfrac{1}{\mu_2^*}\right) - 1$. Now note that $d(x) > 0$ for all $x \in (-\infty, s)$ and is strictly concave (since $\mu_1^* < \mu_2^*$). Hence, we have that $d(0) > \frac{1}{2}d(-1) + \frac{1}{2}d(1)$. Additionally, we obtain from the AM-GM inequality that $\frac{1}{2}d(-1) +$

$\frac{1}{2}d(1) > \sqrt{d(-1)d(1)}$. This in turn implies $d(0) > \sqrt{d(-1)d(1)}$ which is equivalent to the above inequality.

3. This follows from arguments made earlier. The sequence $\{\bar{\mu}_{l-j}^{t-j}\}_{0\leq j\leq l}$ of beliefs arises conditional on good signals arriving and nb being played continuously. Morever, once $\mu_0^{t-l} = \mu_2^*$ is reached, no further information is exchanged. This implies that there can atmost be $l+1$ periods of interactions. Moreover, if $\pi \in [\bar{\pi}_t, 1)$, there will be atmost t periods of interaction.

4. Now consider learning the buyer's type. Suppose $\pi \geq \bar{\pi}_1$. Consider first the case $\pi \in [\bar{\pi}_l, \bar{\pi}_{l+1})$ when $l \geq 1$. If only good signals are received, then beliefs about G will only go upwards and along the sequence $\{\bar{\mu}_{l-j}^{t-j}\}_{0\leq j\leq l}$ as long as nb is being played. Now notice if beliefs go along this sequence but are less than $\bar{\mu}_1^{t-l-1}$, then any play of b at any point would reveal that the buyer is v_1. At the belief equal to $\bar{\mu}_1^{t-l-1}$, buyer v_1 plays b and v_2 plays nb leading to full revelation of buyer types. Hence, the buyer's type is almost surely learnt conditional on good signals always being received. Now consider $\pi \in [\bar{\pi}_t, 1)$. If only good signals arrive, then beliefs about ω always stay at μ_1^*. Buyer v_1 will however keep randomizing leading to the sequence of beliefs $\{\bar{\pi}_k\}_{0\leq k\leq t}$ about buyer types. If at any stage $\pi > \bar{\pi}_1$ a choice of b reveals the buyer to be v_1. If nb is played throughout, then after $t-1$ stages the belief will be $\pi = \bar{\pi}_1$ and only one period remaining. At $(\mu_1^*, \bar{\pi}_1, 1)$, buyer 1 woud play b and buyer 2 would play nb leading to full revelation again. Further note that we established a claim stronger than the one made. From part 2) we know that in the good state G, only good signals arrive. However, in the bad state B, the good signals can still arrive with strictly positive probability.

7.5 Proof of Proposition 6

We prove the two statements made :

(a) We know from the strategies that for any T-period game, if $\pi_0 \in [\bar{\pi}_l, \bar{\pi}_{l+1})$ for $0 \leq l < T$, the experiment chosen in the first period is equal to $(0, \bar{\mu}_l^T)$. Now given $\epsilon > 0$, we define $\bar{T}$ as follows. Since we know $\lim_{t\to\infty} \bar{\mu}_l^t = \mu_2^*$ for every $l \geq 0$. Now we also now that $\bigcup_{k\geq 0}[\bar{\pi}_k, \bar{\pi}_{k+1}) = [0,1)$. First, we define $\hat{L} \in \mathbb{N}$ to be such that $\pi_0 \in [\bar{\pi}_{\hat{L}}, \bar{\pi}_{\hat{L}+1})$. Then we define $\bar{T}$ to be such that $\bar{T} > \hat{L}$ and satisfies

$$|\bar{\mu}_{\hat{L}}^{\bar{T}} - \mu_2^*| < \epsilon$$

Now consider any $T \geq \bar{T}$. We know that in the game $\hat{G}^T_(\mu_0, \pi_0)$, the experiment $(0, \bar{\mu}^{\bar{T}}_{\hat{L}})$ is chosen in the beginning of the game. Now, since $(0, \bar{\mu}^{\bar{T}}_{\hat{L}})$. Since $\bar{\mu}^t_l$ is increasing in t, it follows that $|\bar{\mu}^T_{\hat{L}} - \mu^*_2| = \mu^*_2 - \bar{\mu}^T_{\hat{L}} \leq \mu^*_2 - \bar{\mu}^{\bar{T}}_{\hat{L}} = |\bar{\mu}^{\bar{T}}_{\hat{L}} - \mu^*_2| < \epsilon$. We also know that conditional on a_1 continually being played the sequence of future experiments will be $\{(0, \bar{\mu}^{\bar{T}-j}_{\hat{L}-j})\}_{1 \leq j \leq \hat{L}}$. Note that $\bar{\mu}^{\bar{T}}_{\hat{L}} \leq \bar{\mu}^{\bar{T}-j}_{\hat{L}-j} \leq \mu^*_2$. Hence, future posteriors are either equal to 0 or lie in $[\bar{\mu}^T_{\hat{L}}, \mu^*_2]$.

(b) The argument is analogous to a). We first obtain $\bar{L} \in \mathbb{N}$ such that $\pi_0 \in [\bar{\pi}_{\bar{L}}, \bar{\pi}_{\bar{L}+1})$. We then define $\bar{T}$ such that $1 - \epsilon < \bar{\mu}^{\bar{T}}_{\bar{L}} < 1$. Since $\bar{\mu}^t_l$ is increasing in t, we get $\{\bar{\mu}^T_{\bar{L}}\}_{T \geq \bar{T}} \subseteq (1 - \epsilon, 1)$. This implies for $\hat{G}^T(\mu_0, \pi_0)$, the experiment chosen in the first round is close to full disclosure.

7.6 Only thresholds matter

We focus attention on state dependent utility functions $u : A \times \Omega \to \mathbb{R}$ for which it is true that $u(a_d, G) > u(a_s, G)$ and $u(a_s, B) > u(a_s, B)$. Call any utility function u *regular* if it satisfies the condition . Clearly for any regular function there exists $\mu^* \in (0, 1)$ such that a_d gives strictly higher expected utility for any $\mu > \mu^*$ and a_s gives strictly higher expected utility for any $\mu < \mu^*$. At $\mu = \mu^*$, both a_d and a_s give the same expected utility. Call μ^* the *threshold* associated with the function u. We now provide a proof of Claim 2 :

Claim 2 : Let u, u' be two state dependent utility functions with the same threshold. There exists a $\kappa > 0$ and $c : \Omega \to \mathbb{R}$ such that

$$u'(a, \omega) = \kappa u(a, \omega) + c(\omega)$$

Proof. Define the following points in $\mathbb{R}^2$: $u_d := (u(a_d, G), u(a_d, B))$, $u_s := (u(a_s, G), u(a_s, B))$, $u'_d := (u'(a_d, G), u'(a_d, B))$ and $u'_s := (u'(a_s, G), u'(a_s, B))$. By hypothesis, we know there exists a unique $m^* := (\mu^*, 1 - \mu^*) \in \Delta(\Omega)$ such that $m^*.(u_d - u_s) = m^*.(u'_d - u'_s) = 0$. Define $x := u_d - u_s$ and $x' := u'_d - u'_s$. Clearly x, x' both belong to the nullspace of m^* which has dimension equal to 1. Hence, there exists $\kappa \in \mathbb{R}$ such that $x' = \kappa x$. Since u and u' are both regular it follows that $\kappa > 0$. Now define $c = (c(G), c(B)) := u'_s - \kappa u_s$. This implies $u'_d = \kappa u_d + c$ and $u'_s = \kappa u_s + c$ as a result we obtain $u'(a, \omega) = \kappa u(a, \omega) + c(\omega)$. □